Inertial Games Dynamics

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- Weakly dominated strategies may survive (Samuelson 1993).
- Our question: Can second order dynamics be introduced **naturally** in games? Do they have **better** convergence properties?

Model and Notation

Basic ingredient: a multi-player game in normal form $\mathfrak{G} \equiv \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$:

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Special case: if $u_{k\alpha}(x) - u_{k\beta}(x) = -\left[V(\alpha; x_{-k}) - V(\beta; x_{-k})\right]$ for some $V \colon X \to \mathbb{R}$, the game is called a potential game.

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Equilibrium: we will say that $q \in X$ is a Nash equilibrium of $\mathfrak{G}$ if

$$u_{k\alpha}(q) \geq u_{k\beta}(q) \text{ for all } \alpha \in \operatorname{supp}(q_k),\ \beta \in \mathcal{A}_k,\ k \in \mathcal{N}.$$

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One of the most widely studied dynamics is the replicator equation (RD).

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- Strict Nash equilibria and ESS are asymptotically stable.
- In potential games, global converge to the set of Nash equilibria.
- BUT: weakly dominated strategies may survive, and all strategies in the support of the unique correlated equilibria may be eliminated (Viossat).
- Question: Could a "second order replicator dynamics" improve the rationality properties?

Reinforcement Learning Loop

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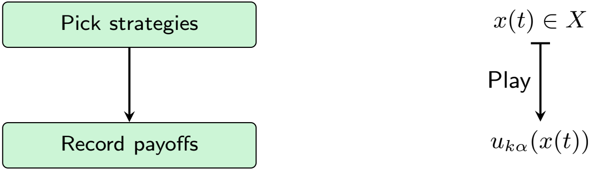
Pick strategies

$x(t) \in X$

Reinforcement Learning Loop



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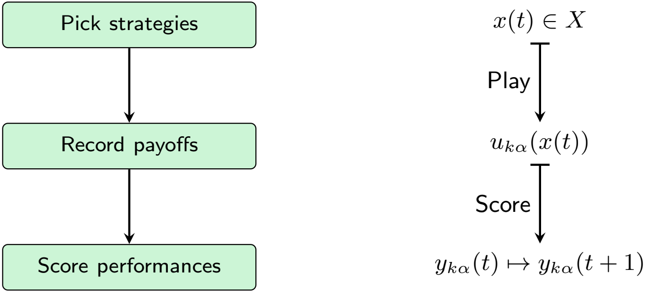

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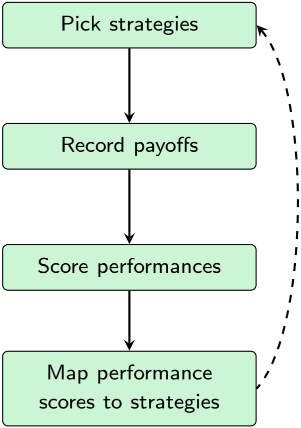

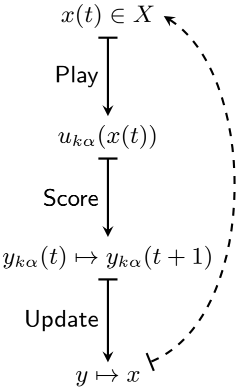

Reinforcement Learning: Updating Scores

A simple updating rule: score's rate of change is the instantaneous payoff:

$$\dot{y}_{k\alpha}(t) = u_{k\alpha}(x(t))$$

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Coupled with the Gibbs map $x_{k\alpha} = \exp(y_{k\alpha}) \Big/ \sum_{\beta}^{k} \exp(y_{k\beta})$, this updating rule yields the (first-order) replicator dynamics:

$$\dot{x}_{k\alpha} = x_{k\alpha} \left(u_{k\alpha}(x) - \sum\nolimits_{\beta}^{k} x_{k\beta} u_{k\beta}(x) \right)$$

(Hofbauer et al. 09; Mertikopoulos-Moustakas 10; Rustichini 99; Sorin, 09)

Reinforcement Learning: Second Order Effects

What if the rate of change corresponds to the cumulative payoff ?

$$\dot{y}_{k\alpha} = U_{k\alpha}$$

where $U_{k\alpha}(t) = \int_0^t u_{k\alpha}(x(s))ds$ is the cumulative payoff of strategy α.

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Coupled with the Gibbs map, we obtain the *second order replicator dynamics*:

$$\ddot{x}_{k\alpha} = x_{k\alpha}\left(u_{k\alpha}(x) - u_k(x)\right) + x_{k\alpha}\left(\dot{x}_{ka}^2/x_{k\alpha}^2 - \sum_\beta^k \dot{x}_{k\beta}^2/x_{k\beta}\right)$$

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Observations: the dynamics is well defined and the initial velocity $\dot{y}(0)$ is 0.

Iterative Extinction of Strictly Dominated Strategies

Theorem

Let $x(t)$ be an interior solution path of second order replicator dynamics. If $q_k \in X_k$ is iteratively strictly dominated, then:

$$D_{\mathrm{KL}}(q_k \,\|\, x_k(t)) \geq \lambda_k c t^2/2 + \mathcal{O}(t),$$

where $c > 0$ and D_{KL} is the K-L divergence
$D_{\mathrm{KL}}(q_k \,\|\, x_k) = \sum_{\alpha}^{k} q_{k\alpha} \log\left(q_{k\alpha}/x_{k\alpha}\right)$.

In particular, for pure strategies $\alpha \prec \beta$, we have:

$$x_{k\alpha}(t)/x_{k\beta}(t) \leq \exp\left(-\lambda_k \Delta u_{\beta\alpha} t^2/2 + \mathcal{O}(t)\right),$$

where $\Delta u_{\beta\alpha} = \min_{x \in X}\{u_{k\beta}(x) - u_{k\alpha}(x)\} > 0$.

In words: iteratively strictly dominated strategies become extinct in the second order replicator dynamics.

Elimination of Weakly Dominated Strategies

Theorem

Let $x(t)$ be an interior solution orbit of the n-th order ($n \geq 2$) replicator dynamics that starts at rest: $\dot{x}(0) = 0$.

If $q_k \in X_k$ is weakly dominated, then it becomes extinct along $x(t)$ at a rate

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where λ_k is the learning rate of player k and $c > 0$ is a positive constant.

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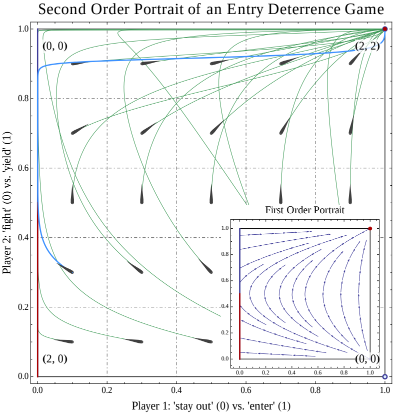
- In word: weakly dominated strategies extinct if players start unbiased.
- No extension for iteratively weakly dominated strategies.

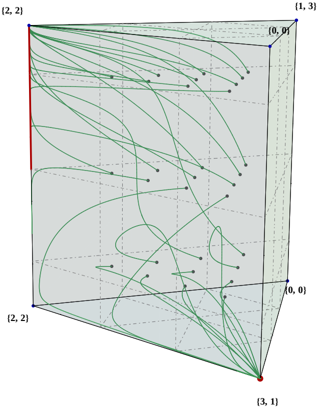
Weakly Dominated Strategies

Weakly dominated strategies in the second order replicator dynamics:

Entry Deterrence



Outside Option



A Second Order Folk Theorem

Theorem

Let $x(t)$ be a solution orbit of the n-th order replicator dynamics. Then:

I. $x(t) = q$ *for all* $t \geq 0$ *if and only if* q *is a restricted equilibrium of* $\mathfrak{G}$.

II. *If* $x(0) \in \text{int}(X)$ *and* $\lim_{t\to\infty} x(t) = q$, *then* q *is a Nash equilibrium of* $\mathfrak{G}$.

III. *If every neighborhood* U *of* q *in* X *admits an interior orbit* $x_U(t)$ *such that* $x_U(t) \in U$ *for all* $t \geq 0$, *then* q *is a Nash equilibrium of* $\mathfrak{G}$.

IV. q *is a strict equilibrium if and only if* $\lim_{t\to\infty} x(t) = q$ *for all trajectories that starts at rest in a neighborhood of* $x(0)$.

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- ESS cannot be asymptotically stable under this dynamics.
- No obvious law of dissipation of energy property in potential game, necessary to obtain global convergence to Nash.

The Gradient Ascent Interpretation of RD

- The first order replicator dynamics can be seen as a gradient ascent scheme with respect to a Riemannian Metric, called the Shahshahani metric.

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- This also implies that in convex optimisation, this dynamics define a interior gradient ascent method that globally converges to the global minimum (Alvarez, Brahic and Bolte).
- The extension to second order is called the heavy ball with friction.

The Heavy Ball with Friction without constraints

The heavy ball with friction dynamics (Attouch, Goudou and Redont) on $\mathbb{R}^m$ are:

$$\ddot{x} = -\nabla V - \eta \dot{x}, \tag{HBF}$$

where $V : \mathbb{R}^m \to \mathbb{R}$ is a smooth potential function and $\eta > 0$ is the friction coefficient which dissipates energy.

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To apply the method to a constraint space X we need to:

1. Endow X with a Hessian Riemannian structure (following the first order approach of Alvarez, Brahic and Bolte).
2. Derive the Riemannian analogue of (HBF).

Riemannian Metrics

A Riemannian metric on an open set $U \subseteq \mathbb{R}^m$ is a smoothly varying scalar product on U

$$g(X,Y) \equiv \langle X,Y\rangle_g = \sum\nolimits_{j,k} X_j g_{jk} Y_k, \qquad X,Y \in \mathbb{R}^m,$$

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A Riemannian metric generates the notion of covariant differentiation along a curve, so that, the acceleration of $x(t)$ is:

$$\frac{D^2 x_k}{Dt^2} = \ddot{x}_k + \sum_{i,j} \Gamma^k_{ij} \dot{x}_i \dot{x}_j.$$

where:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{-1}_{k\ell} \left(\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}\right)$$

Hessian Riemannian Metrics

We will be interested in a specific class of Riemannian metrics on $\mathbb{R}^n_{++}$ generated by a family of barrier functions.

Definition
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- Euclidean metric (non-example): $\theta(x) = \frac{1}{2}x^2 \implies g_{ij}(x) = \delta_{ij}$.
- Shahshahani metric: $\theta(x) = x \log x \implies g_{ij}(x) = \delta_{ij}/x_j$.
- Log-barrier metric: $\theta(x) = -\log x \implies g_{ij}(x) = \delta_{ij}/x_j^2$.

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$$\frac{D^2 x}{Dt^2} = -\nabla_g V - \eta \dot{x},$$

Leading to the inertial game dynamics:

$$\begin{aligned} \ddot{x}_{k\alpha} = & \frac{1}{\theta''_{k\alpha}} \left[u_{k\alpha} - \sum\nolimits_{\beta} \left(\Theta''_{k,h} / \theta''_{k\beta} \right) u_{k\beta} \right] \\ & - \frac{1}{2} \frac{1}{\theta''_{k\alpha}} \left[\theta'''_{k\alpha} \dot{x}^2_{k\alpha} - \sum\nolimits_{\ell} \left(\Theta''_{k,h} / \theta''_{k\beta} \right) \theta'''_{k\beta} \dot{x}^2_{k\beta} \right] - \eta \dot{x}_{k\alpha}, \end{aligned} \tag{IGD}$$

1. The Gibbs kernel $\theta(x) = x \log x$ generates the inertial replicator dynamics:

$$\ddot{x}_{k\alpha} = x_{k\alpha}\left(u_{k\alpha} - \sum\nolimits_{\beta} x_{k\beta}u_{k\beta}\right) + \frac{1}{2}x_{k\alpha}\left(\dot{x}_{k\alpha}^2/x_{k\alpha}^2 - \sum\nolimits_{\beta} \dot{x}_{k\beta}^2/x_{k\beta}\right) - \eta\dot{x}_{k\alpha}. \tag{I-RD}$$

2. The Burg kernel $\theta(x) = -\log x$ generates the inertial log-barrier dynamics:

$$\ddot{x}_{k\alpha} = x_{k\alpha}^2\left(u_{k\alpha} - r_k^{-2}\sum\nolimits_{\beta} x_{k\beta}^2 u_{k\beta}\right) + x_{k\alpha}^2\left(\dot{x}_{k\alpha}^2/x_{k\alpha}^3 - r_k^{-2}\sum\nolimits_{\beta} \dot{x}_{k\beta}^2/x_{k\beta}\right) - \eta\dot{x}_{k\alpha} \tag{I-LD}$$

where $r_k^2 = \sum_\beta x_{k\beta}^2$.

Potential Games

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Theorem

Assume that the inertial dynamics are well-posed, and let q be a local minimizer of V with Hess $(V) \succ 0$ at q. If $x(0)$ is sufficiently close to q and the system's initial kinetic energy $K(0) = \frac{1}{2}\|\dot{x}(0)\|_g^2$ is low enough, then $\lim_{t\to\infty} x(t) = q$.

Multi-player games

In our inertial setting, we have the following folk-type theorem:

Theorem

Assume that the inertial dynamics are well-posed, and let $x(t)$ be a solution orbit. Then:

I. $x(t) = q$ *for all* $t \geq 0$ *if and only if* q *is a restricted equilibrium.*

II. *If* $x(t)$ *is interior and* $\lim_{t\to\infty} x(t) = q$*, then* q *is a restricted equilibrium.*

III. *If every neighborhood* U *of* q *in* X *admits an interior orbit* $x_U(t)$ *such that* $x_U(t) \in U$ *for all* $t \geq 0$*, then* q *is a restricted equilibrium.*

IV. *If* q *is a strict equilibrium of* $\mathfrak{G}$ *and* $x(t)$ *starts close enough to* q *with sufficiently low speed* $\|\dot{x}(0)\|_g$*, then* $x(t)$ *remains close to* q *for all* $t \geq 0$ *and* $\lim_{t\to\infty} x(t) = q$.

An Isometric Embedding into Euclidean Space

The above results all rely on the inertial dynamics being well-posed.

Proposition (Nash embedding)

Let $\xi_\alpha = \phi(x_\alpha)$ *with* $\phi'(x) = \sqrt{\theta''(x)}$, *and set*

$$S = \{(\phi(x_0), \ldots, \phi(x_n)) : x \in int(\Delta)\}.$$

Then S *with the euclidean metric of* $\mathbb{R}^{n+1}$ *is isomorphic to the open unit simplex with the Riemannian metric generated by* θ*.*

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2. The open unit simplex with the log-barrier metric $g_{ij} = \delta_{ij}/x_j^2$ is isometric to the closed hypersurface $S = \{\xi \in \mathbb{R}^{n+1} : \xi_\alpha < 0 \text{ and } \sum_\beta e^{\xi_\beta} = 1\}$.

Well-posedness of the Inertial Dynamics

$$S = \{\xi \in \mathbb{R}^{n+1} : \sum_{\beta} \phi^{-1}(\xi_\beta) = 1\}$$

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The inertial dynamics is well-posed if and only if S is a closed hypersurface.

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Proof:

In the Euclidean variables $\xi = \phi(x)$, the inertial dynamics is just Newton's ordinary second law of motion for particles constrained to move on S.
If S is bounded in some direction, then orbits can escape from that part of S in finite time.

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If the answer is yes, then, the optimization approach leads to a class of dynamics that strictly improves the RD rationality properties.