The convergence problem in Mean Field Games

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Mean Field Games (MFG) study

- nonatomic games = infinitely many agents having individually a negligible influence on the global system
  (Ref : Aumann ('64), Schmeidler ('73), Hildenbrand ('74), Mas-Colell ('84), ...)

- in a dynamical framework = each agent acts on his state which evolves in continuous time and has a payoff depending on the other's position
  (related to optimal control)

Some references :

- Early work by Lasry-Lions (2006) and Huang-Caines-Malhamé (2006)
- Similar models in the economic literature : heterogeneous agent models
  (Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

Statement of the problem

Analyse the limit of the $N-$player differential game as $N \to +\infty$.

Motivation :

- derive collective behavior from individual ones
  More precisely, obtain heterogeneous agent models from Nash equilibria.
- Inspired by mean field models in mathematical physics.
- Continuous-time version of Hildebrand (1974) “Core and Equilibria of a Large Economy".

Main difficulty :

- In the Nash system, players observe each other ;
  in the limit game, players observe only the evolving density of the other agents.

Description of the $N-$player game

- $N-$players.
- Player i (for $i = 1, \dots, N$) acts through his control $(\alpha_{i,t})$ on his state $(X_{i,t})$, which evolves according to the SDE

$$dX_{i,t} = \alpha_{i,t}dt + \sqrt{2}dB_t^i + \sqrt{2\beta}dW_t, \qquad X_{t_0} = x_{i,0},$$

  where $(B_t^i)_{t\in[0,T]}$ and $(W_t)_{t\in[0,T]}$ are independent BM.
- The (B_t^i) correspond to the *individual noise* (specific to each player), (W_t) being the *common noise*, affecting all the players.
- Player aims at minimizing the cost function J_i^N, given, if the initial position of the system is $\boldsymbol{x}_0 = (x_{1,0}, \dots, x_{N,0})$ at time t_0, by

$$J_i^N(t_0, \boldsymbol{x}_0, (\alpha_j)) = \mathbb{E}\left[\int_{t_0}^T \left(L(X_{i,s}, \alpha_{i,s}) + F^{N,i}(\boldsymbol{X}_s)\right) ds + G^{N,i}(\boldsymbol{X}_T)\right]$$

  where $\boldsymbol{X}_t = (X_{1,t}, \dots, X_{n,t})$ and where $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $F^{N,i} : \mathbb{R}^{Nd} \to \mathbb{R}$ and $G^{N,i} : \mathbb{R}^{Nd} \to \mathbb{R}$ are are given maps.

- The Hamiltonian of the problem is :

$$\forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad H(x,p) = \sup_{\alpha \in \mathbb{R}^d} \{-\alpha \cdot p - L(x,\alpha)\}.$$

- Perfect Nash equilibria in Markov strategies are given by the system Nash system :

$$\left\{ \begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^N \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^N \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) \\ \qquad + \sum_{j \neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = F^{N,i}(\boldsymbol{x}) \\ \qquad\qquad \text{in } [0,T] \times (\mathbb{R}^d)^N, \ i \in \{1,\dots,N\} \\ v^{N,i}(T,\boldsymbol{x}) = G^{N,i}(\boldsymbol{x}) \qquad \text{in } (\mathbb{R}^d)^N, \ i \in \{1,\dots,N\} \end{array} \right.$$

- If $(v^{N,i})_{i=1,\dots,N}$ is the solution of the Nash system, then the feedback strategies

$$\left(\alpha_i^*(t,\boldsymbol{x}) := -D_p H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x}))\right)_{i=1,\dots,N}$$

provide a feedback Nash equilibrium for the game :

$$v^{N,i}(t_0,\boldsymbol{x}_0) = J_i^N(t_0,\boldsymbol{x}_0,(\alpha_j^*)_{j=1,\dots,N}) \leq J_i^N(t_0,\boldsymbol{x}_0,\alpha_i,(\hat{\alpha}_j^*)_{j\neq i})$$

for any $i \in \{1,\dots,N\}$ and any control α_i adapted to the filtrations generated by $(B^j)_{j=1,\dots,N}$ and W.

The system of optimal trajectories.

We denote by $\boldsymbol{X}_t^N = (X_{1,t}^N, \dots, X_{N,t}^N)$ the "optimal trajectories" of the $N-$player game : they solve the system of N coupled stochastic differential equations (SDE) :

$$dX_{i,t}^N = -D_pH(X_{i,t}, Dv^{N,i}(t, \boldsymbol{X}_t^N))dt + \sqrt{2}dB_t^i + \sqrt{2\beta}dW_t, \qquad t \in [0,T],\ i \in \{1, \dots, N\},$$

where

- $v^{N,i}$ is the solution to the Nash system,
- the $(B_t^i)_{t\in[0,T]}$ and $(W_t)_{t\in[0,T]}$ are $d-$dimensional independent Brownian motions.

Problem : Analyse the limit of the $v^{N,i}$ and of the $(X_{i,t}^N)$ as $N \to +\infty$.

The symmetry assumption

To expect a limit, we suppose that, in the differential games, players are indistinguishable.

Namely, we assume that the $F^{N,i}$ and $G^{N,i}$ are of the form :

$$F^{N,i}(\boldsymbol{x}) = F(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{and} \qquad G^{N,i}(\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}),$$

where

- $m_{\boldsymbol{x}}^{N,i} = \dfrac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$ is the empirical distribution of the $(x_j)_{j \neq i}$,
- $F, G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are “smooth functions"
  ($\mathcal{P}(\mathbb{R}^d)$ being the set of Borel probability measures on $\mathbb{R}^d$.)

Under this assumption,

- the solution of the Nash system enjoy strong symmetry properties,
- the $(X_{i,\cdot}^N)$ are exchangeable (if this is the case for the $(X_{i,0}^N)$).

Summary of the problem

Describe the behavior, as N tends to infinity,

- of the Nash system

$$\left\{\begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^{N} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) \\ \qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = F(x_i, m_{\boldsymbol{x}}^{N,i}) \\ \qquad\qquad \text{in } [0,T] \times (\mathbb{R}^d)^N \\ v^{N,i}(T,\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } (\mathbb{R}^d)^N \end{array}\right.$$

- and of the associated optimal trajectories

$$dX_{i,t}^N = -D_p H(X_{i,t}, Dv^{N,i}(t,\boldsymbol{X}_t^N))dt + \sqrt{2}dB_t^i + \sqrt{2\beta}dW_t, \qquad t \in [0,T],\ i \in \{1,\dots,N\},$$

The MFG system.

- In the limit as $N \to +\infty$ and for $\beta = 0$, the players should take into account only the evolving density of the other players. This formally yields to the MFG system :

$$(MFG) \qquad \left\{ \begin{array}{ll} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) & \text{in } [0,T]\times\mathbb{R}^d \\ \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ u(T,x) = G(x, m(T)), \ m(t_0, \cdot) = m_0 & \text{in } \mathbb{R}^d \end{array} \right.$$

- Introduced by Lasry-Lions and by Huang-Caines-Malhamé and studied intensively since then.
- Existence and uniqueness of solution under a monotonicity assumptions for F and G (here for F) :

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))d(m-m')(x) \geq 0.$$

  This assumption “means" that the players dislike congested areas. (Lasry-Lions, 2006)
- Optimal strategies associated with the MFG system yield to $\varepsilon-$optimal open-loop strategies in the $N-$player game.
  (Huang-Caines-Malhamé 2006, Carmona-Delarue 2012, Kolokoltsov 2014).

The master equation.

- Because of the symmetry, the $v^{N,i}$ can be written as

$$v^{N,i}(t,\boldsymbol{x}) = U^N(t, x_i, m_{\boldsymbol{x}}^{N,i}).$$

- Following Lasry-Lions, the expected limit U of the (U^N) should formally satisfy the master equation.

$$\left\{\begin{array}{l} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m U\right]\, dm(y) + \int_{\mathbb{R}^d} D_m U\cdot D_p H(y, D_x U)\, dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\left[D_m U\right] dm(y) - \beta\int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right]\, dm\otimes dm = F(x,m) \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \end{array}\right.$$

where

- the unknown is $U : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}$,
- $\partial_t U$, $D_x U$ and $\Delta_x U$ stand for the usual derivatives with respect to the local variables (t,x) of U,
- $D_m U$ and $D^2_{mm} U$ are the first and second order derivatives with respect to the measure m.

But so far :

- No link between the solutions $(v^{N,i})$ of the Nash system
(open-loop regime studied by Fischer ('14) and by Lacker ('14)).

- Little rigorous results on the master equations
(Lions ('13), Buckdahn-Li-Peng-Rainer ('14), Gangbo-Swiech ('14), Chassagneux-Crisan-Delarue ('15), Bessi ('15))

Our contribution :

- Well-posedness of the master equation,
(when the ambiant space is the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$)

- Limit results for the Nash system and the associated optimal trajectories,
(using the well-posedness of the master equation in a key way)

Statement of the results

Standing assumptions :

- Ambient space $\mathbb{T}^d$,
- H smooth and $D^2_{pp}H > 0$,
- F and G smooth and monotone.

Theorem 1

Under our standing assumptions, the **master equation** has a unique classical solution.

Theorem 2

Let $(v^{N,i})$ be the solution to the Nash system and U be the classical solution to the **master equation**. For any $N \geq 1$, $\boldsymbol{x} \in (\mathbb{T}^d)^N$, let $m^N_{\boldsymbol{x}} := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. Then

$$\frac{1}{N}\sum_{i=1}^N \left| v^{N,i}(t_0,\boldsymbol{x}) - U(t_0, x_i, m^N_{\boldsymbol{x}}) \right| \leq CN^{-1/2}.$$

Let $t_0 \in [0,T)$, $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and let (Z_i) be an i.i.d family of random variables of law m_0. Let also (B^i) and W be independent B.M. and independent of the (Z_i).

We consider

- the optimal trajectories $(\boldsymbol{X}_t^N = (X_{1,t}^N, \dots, X_{N,t}^N))_{t\in[t_0,T]}$ of the Nash system :

$$\begin{cases} dX_{i,t}^N = -D_pH(X_{i,t}^N, D_{x_i}v^{N,i}(t,\boldsymbol{X}_t^N))dt + \sqrt{2}dB_t^i + \sqrt{2\beta}dW_t, \qquad t\in[t_0,T] \\ X_{i,t_0}^N = Z_i \end{cases}$$

- and the solution $(\boldsymbol{Y}_t^N = (Y_{1,t}^N, \dots, Y_{N,t}^N))_{t\in[t_0,T]}$ of stochastic differential equation of McKean-Vlasov type :

$$\begin{cases} dY_{i,t}^N = -D_pH\left(Y_{i,t}^N, D_xU(t, Y_{i,t}^N, \mathcal{L}(Y_{i,t}^N|W)\right) dt + \sqrt{2}dB_t^i + \sqrt{2\beta}dW_t, \\ Y_{i,t_0}^N = Z_i. \end{cases}$$

Theorem 3

For any $N \geq 1$ and any $i \in \{1,\dots,N\}$, we have

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \left|X_{i,t}^N - Y_{i,t}^N\right|\right] \leq CN^{-\frac{1}{d+8}}$$

for some constant $C > 0$ independent of t_0, m_0 and N.

Conclusion

We have established

- the well-posedness of the master equation,
- limit results for the Nash system and the associated optimal trajectories.

Open problems :

- Analysis in more realistic setting (with boundary conditions, non constant diffusion matrices,...)
- Stronger convergence for the solutions $(v^{N,i})$ of the Nash system.
- Strongly interacting systems (when F and G are *local couplings*).