Existence of Nash Equilibria in Games with Incomplete Preferences.

Philippe Bich (Paris 1 Panthéon-Sorbonne and PSE)
Rida Laraki (CNRS, Lamsade, Paris Dauphine, and Economics Department of Ecole Polytechnique)

Paris, 27 Mai 2015.

1) Definitions of the class of games

We first define the class of games we study.

1) a) Strategic Game

A strategic game is $((X_i)_{i \in N}, (u_i)_{i \in N})$ where

- N is the finite set of players, and for every $i \in N$:
- X_i is the strategy space of player i (convex, compact subset of a metric space)
- $u_i : X = \prod_{i \in N} X_i \to \mathbf{R}$ is the payoff function of player i.

1) a) Strategic Game

A strategic game is $((X_i)_{i \in N}, (u_i)_{i \in N})$ where

- N is the finite set of players, and for every $i \in N$:
- X_i is the strategy space of player i (convex, compact subset of a metric space)
- $u_i : X = \prod_{i \in N} X_i \to \mathbf{R}$ is the payoff function of player i.

1) a) Strategic Game

A strategic game is $((X_i)_{i\in N}, (u_i)_{i\in N})$ where

- N is the finite set of players, and for every $i \in N$:
- X_i is the strategy space of player i (convex, compact subset of a metric space)
- $u_i : X = \prod_{i\in N} X_i \to \mathbf{R}$ is the payoff function of player i.

1) a) Strategic Game

A strategic game is $((X_i)_{i \in N}, (u_i)_{i \in N})$ where

- N is the finite set of players, and for every $i \in N$:
- X_i is the strategy space of player i (convex, compact subset of a metric space)
- $u_i : X = \prod_{i \in N} X_i \to \mathbf{R}$ is the payoff function of player i.

1) b) Strategic Game with endogenous sharing rule

Add an ingredient: allow the payoffs to be multivalued (see Simon-Zame, Econometrica (1990)):

1) b) Strategic Game with endogenous sharing rule

A strategic game with endogenous sharing rule is a couple $((X_i)_{i \in N}, \mathcal{U})$ where:

$\mathcal{U}$ is a multi-valued function from $X := \prod_{i \in N} X_i$ to $\mathbf{R}^N$ with non empty values

Interpretation: $\mathcal{U}(x) \subset \mathbf{R}^N$ is the universe of all possible payoff profiles at $x \in X$.

In this talk, we shall be interested in the existence of a selection u of $\mathcal{U}$ for which there is a pure-strategy Nash equilibrium.

1) b) Strategic Game with endogenous sharing rule

A strategic game with endogenous sharing rule is a couple $((X_i)_{i \in N}, \mathcal{U})$ where:

$\mathcal{U}$ is a multi-valued function from $X := \prod_{i \in N} X_i$ to $\mathbf{R}^N$ with non empty values

Interpretation: $\mathcal{U}(x) \subset \mathbf{R}^N$ is the universe of all possible payoff profiles at $x \in X$.

In this talk, we shall be interested in the existence of a selection u of $\mathcal{U}$ for which there is a pure-strategy Nash equilibrium.

1) b) Strategic Game with endogenous sharing rule

A strategic game with endogenous sharing rule is a couple $((X_i)_{i \in N}, \mathcal{U})$ where:

$\mathcal{U}$ is a multi-valued function from $X := \prod_{i \in N} X_i$ to $\mathbf{R}^N$ with non empty values

Interpretation: $\mathcal{U}(x) \subset \mathbf{R}^N$ is the universe of all possible payoff profiles at $x \in X$.

In this talk, we shall be interested in the existence of a selection u of $\mathcal{U}$ for which there is a pure-strategy Nash equilibrium.

1) b) Strategic Game with endogenous sharing rule

A strategic game with endogenous sharing rule is a couple $((X_i)_{i \in N}, \mathcal{U})$ where:

$\mathcal{U}$ is a multi-valued function from $X := \prod_{i \in N} X_i$ to $\mathbf{R}^N$ with non empty values

Interpretation: $\mathcal{U}(x) \subset \mathbf{R}^N$ is the universe of all possible payoff profiles at $x \in X$.

In this talk, we shall be interested in the existence of a selection u of $\mathcal{U}$ for which there is a pure-strategy Nash equilibrium.

1) c) Interpretations of endogenous sharing rule.

3 interpretations of endogenous sharing rule: the two first by Simon-Zame, the last one (incomplete preferences) is ours.

1) c) First Interpretation: indeterminacy of payoffs

- At each profile x, every $(v_1, ..., v_N) \in \mathcal{U}(x)$ specifies for each player i his share v_i.
- In particular, a selection u of $\mathcal{U}$ is a sharing rule.
- The use of a multivalued function is because sharing exhibit some indeterminacy (see location game).

1) c) First Interpretation: indeterminacy of payoffs

- At each profile x, every $(v_1, ..., v_N) \in \mathcal{U}(x)$ specifies for each player i his share v_i.
- In particular, a selection u of $\mathcal{U}$ is a sharing rule.
- The use of a multivalued function is because sharing exhibit some indeterminacy (see location game).

1) c) First Interpretation: indeterminacy of payoffs

- At each profile x, every $(v_1, ..., v_N) \in \mathcal{U}(x)$ specifies for each player i his share v_i.
- In particular, a selection u of $\mathcal{U}$ is a sharing rule.
- The use of a multivalued function is because sharing exhibit some indeterminacy (see location game).

1) c) First Interpretation: indeterminacy of payoffs

- At each profile x, every $(v_1, ..., v_N) \in \mathcal{U}(x)$ specifies for each player i his share v_i.
- In particular, a selection u of $\mathcal{U}$ is a sharing rule.
- The use of a multivalued function is because sharing exhibit some indeterminacy (see location game).

1) c) First Interpretation: indeterminacy of payoffs

- At each profile x, every $(v_1, ..., v_N) \in \mathcal{U}(x)$ specifies for each player i his share v_i.
- In particular, a selection u of $\mathcal{U}$ is a sharing rule.
- The use of a multivalued function is because sharing exhibit some indeterminacy (see location game).

1) c) Second Interpretation: discontinuous games

- This interpretation corresponds to a particular class of endogenous sharing rule.
- Assume there is a payoff-profile function $u = (u_1, ..., u_N)$
- Define
$$\mathcal{U}(x) = \{\lim_{n \to +\infty} u(x^n) : (x^n) \to x\}$$
- i.e., each element of $\mathcal{U}(x)$ is very close to $u(x')$, for some profile x' close to x.
- It's a way to consider trembling-hand strategies.

1) c) Second Interpretation: discontinuous games

- This interpretation corresponds to a particular class of endogenous sharing rule.
- Assume there is a payoff-profile function $u = (u_1, ..., u_N)$
- Define

$$\mathcal{U}(x) = \{\lim_{n \to +\infty} u(x^n) : (x^n) \to x\}$$

- i.e., each element of $\mathcal{U}(x)$ is very close to $u(x')$, for some profile x' close to x.
- It's a way to consider trembling-hand strategies.

1) c) Second Interpretation: discontinuous games

- This interpretation corresponds to a particular class of endogenous sharing rule.
- Assume there is a payoff-profile function $u = (u_1, ..., u_N)$
- Define

$$\mathcal{U}(x) = \{\lim_{n \to +\infty} u(x^n) : (x^n) \to x\}$$

- i.e., each element of $\mathcal{U}(x)$ is very close to $u(x')$, for some profile x' close to x.
- It's a way to consider trembling-hand strategies.

1) c) Second Interpretation: discontinuous games

- This interpretation corresponds to a particular class of endogenous sharing rule.
- Assume there is a payoff-profile function $u = (u_1, ..., u_N)$
- Define

$$\mathcal{U}(x) = \{\lim_{n \to +\infty} u(x^n) : (x^n) \to x\}$$

- i.e., each element of $\mathcal{U}(x)$ is very close to $u(x')$, for some profile x' close to x.
- It's a way to consider trembling-hand strategies.

1) c) Second Interpretation: discontinuous games

- This interpretation corresponds to a particular class of endogenous sharing rule.
- Assume there is a payoff-profile function $u = (u_1, ..., u_N)$
- Define

$$\mathcal{U}(x) = \{\lim_{n \to +\infty} u(x^n) : (x^n) \to x\}$$

- i.e., each element of $\mathcal{U}(x)$ is very close to $u(x')$, for some profile x' close to x.
- It's a way to consider trembling-hand strategies.

1) c) Second Interpretation: discontinuous games

- This interpretation corresponds to a particular class of endogenous sharing rule.
- Assume there is a payoff-profile function $u = (u_1, ..., u_N)$
- Define
$$\mathcal{U}(x) = \{\lim_{n\to+\infty} u(x^n) : (x^n) \to x\}$$
- i.e., each element of $\mathcal{U}(x)$ is very close to $u(x')$, for some profile x' close to x.
- It's a way to consider trembling-hand strategies.

1) c) Third Interpretation: strategic games with incomplete preferences

- "Of all axioms of utility theory, the completeness axiom is perhaps the most questionable" (Aumann.)
- A preorder on X is a reflexive and transitive binary relation.
- Recall every preorder $\lesssim$ on X admits a multi-utility representation (see OK 2007), i.e:

There is a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$.

Thus there is no loss of generality in working with cardinal multi-representations.

1) c) Third Interpretation: strategic games with incomplete preferences

- "Of all axioms of utility theory, the completeness axiom is perhaps the most questionable" (Aumann.)
- A preorder on X is a reflexive and transitive binary relation.
- Recall every preorder $\lesssim$ on X admits a multi-utility representation (see OK 2007), i.e:

There is a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$.

Thus there is no loss of generality in working with cardinal multi-representations.

1) c) Third Interpretation: strategic games with incomplete preferences

- "Of all axioms of utility theory, the completeness axiom is perhaps the most questionable" (Aumann.)
- A preorder on X is a reflexive and transitive binary relation.
- Recall every preorder $\lesssim$ on X admits a multi-utility representation (see OK 2007), i.e:

There is a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$.

Thus there is no loss of generality in working with cardinal multi-representations.

1) c) Third Interpretation: strategic games with incomplete preferences

- "Of all axioms of utility theory, the completeness axiom is perhaps the most questionable” (Aumann.)
- A preorder on X is a reflexive and transitive binary relation.
- Recall every preorder $\lesssim$ on X admits a multi-utility representation (see OK 2007), i.e:

There is a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$.

Thus there is no loss of generality in working with cardinal multi-representations.

1) c) Third Interpretation: strategic games with incomplete preferences

- "Of all axioms of utility theory, the completeness axiom is perhaps the most questionable” (Aumann.)
- A preorder on X is a reflexive and transitive binary relation.
- Recall every preorder $\lesssim$ on X admits a multi-utility representation (see OK 2007), i.e:

There is a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$.

Thus there is no loss of generality in working with cardinal multi-representations.

1) c) Third Interpretation: strategic games with incomplete preferences

- "Of all axioms of utility theory, the completeness axiom is perhaps the most questionable" (Aumann.)
- A preorder on X is a reflexive and transitive binary relation.
- Recall every preorder $\lesssim$ on X admits a multi-utility representation (see OK 2007), i.e:

There is a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$.

Thus there is no loss of generality in working with cardinal multi-representations.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i \in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i \in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i \in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i \in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\precsim_i$ on X as follows:

$$x \precsim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \precsim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i \in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i\in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i \in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

1) c) Third Interpretation: strategic games with incomplete preferences

- Let $((X_i)_{i\in N}, \mathcal{U})$ be a strategic game with endogenous sharing rule.
- For every player i define a preorder $\lesssim_i$ on X as follows:

$$x \lesssim_i y \Leftrightarrow \text{for every selection } u = (u_1, ..., u_N) \text{ of } \mathcal{U},\ u_i(x) \leq u_i(y).$$

- Note $\mathcal{U}_i(x)$ the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-the component.
- Equivalently, $x \lesssim_i y \Leftrightarrow \sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$.
- Means: y is not worse than x whatever the indeterminacy of payoffs.
- In general, incomplete relation.

Example 1

- Consider a strategic game $((X_i)_{i\in N},(u_i)_{i\in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x,\varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow \sup_{\|x'-x\|\leq\varepsilon} u_i(x') \leq \inf_{\|y'-y\|\leq\varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow \sup_{\|x' - x\| \leq \varepsilon} u_i(x') \leq \inf_{\|y' - y\| \leq \varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow \sup_{\|x'-x\| \leq \varepsilon} u_i(x') \leq \inf_{\|y'-y\| \leq \varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow \sup_{\|x'-x\| \leq \varepsilon} u_i(x') \leq \inf_{\|y'-y\| \leq \varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow$ $\sup_{\|x'-x\| \leq \varepsilon} u_i(x') \leq \inf_{\|y'-y\| \leq \varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i\in N}, (u_i)_{i\in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x,\varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow \sup_{\|x'-x\|\leq\varepsilon} u_i(x') \leq \inf_{\|y'-y\|\leq\varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i\in N}, (u_i)_{i\in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow$ $\sup_{\|x'-x\|\leq\varepsilon} u_i(x') \leq \inf_{\|y'-y\|\leq\varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow \sup_{\|x'-x\| \leq \varepsilon} u_i(x') \leq \inf_{\|y'-y\| \leq \varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 1

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.
- Let $\varepsilon > 0$.
- Define $\mathcal{U}(x) = u(B(x, \varepsilon))$.
- The preorder induced is $x \lesssim_i y \Leftrightarrow$ $\sup_{\|x' - x\| \leq \varepsilon} u_i(x') \leq \inf_{\|y' - y\| \leq \varepsilon} u_i(y')$.
- Even when only one player $i = 1$, in general no maximal element for $\lesssim_1$.
- Remark that the orders induced by u are completions of the order induced by $\lesssim_i$, in the sense that:
- $x \lesssim_i y \Rightarrow u_i(x) \leq u_i(y)$.
- In general, we could ask the existence of a completion for which there is a maximal element.

Example 2

- Consider a strategic game $((X_i)_{i\in N}, (u_i)_{i\in N})$, where u is $\varepsilon-$continuous.
- Define $\mathcal{U}(x) = \{\lim_{n\to+\infty} u(x^n) : (x^n) \to x\}$
- The preorder induced is $x \lesssim_i y \Leftrightarrow \limsup_x u_i(x) \leq \liminf_y u_i(y)$.
- Remark that every Nash of some selection of $\mathcal{U}(x)$ will be an $2\varepsilon-$Nash of u.

Example 2

- Consider a strategic game $((X_i)_{i\in N}, (u_i)_{i\in N})$, where u is $\varepsilon-$continuous.
- Define $\mathcal{U}(x) = \{\lim_{n\to+\infty} u(x^n) : (x^n) \to x\}$
- The preorder induced is $x \lesssim_i y \Leftrightarrow \limsup_x u_i(x) \leq \liminf_y u_i(y)$.
- Remark that every Nash of some selection of $\mathcal{U}(x)$ will be an $2\varepsilon-$Nash of u.

Example 2

- Consider a strategic game $((X_i)_{i\in N}, (u_i)_{i\in N})$, where u is $\varepsilon-$continuous.
- Define $\mathcal{U}(x) = \{\lim_{n\to+\infty} u(x^n) : (x^n) \to x\}$
- The preorder induced is $x \lesssim_i y \Leftrightarrow \limsup_x u_i(x) \leq \liminf_y u_i(y)$.
- Remark that every Nash of some selection of $\mathcal{U}(x)$ will be an $2\varepsilon-$Nash of u.

Example 2

- Consider a strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$, where u is $\varepsilon-$continuous.
- Define $\mathcal{U}(x) = \{\lim_{n \to +\infty} u(x^n) : (x^n) \to x\}$
- The preorder induced is $x \lesssim_i y \Leftrightarrow \limsup_x u_i(x) \leq \liminf_y u_i(y)$.
- Remark that every Nash of some selection of $\mathcal{U}(x)$ will be an $2\varepsilon-$Nash of u.

Example 2

- Consider a strategic game $((X_i)_{i\in N}, (u_i)_{i\in N})$, where u is $\varepsilon-$continuous.
- Define $\mathcal{U}(x) = \{\lim_{n\to+\infty} u(x^n) : (x^n) \to x\}$
- The preorder induced is $x \lesssim_i y \Leftrightarrow \limsup_x u_i(x) \leq \liminf_y u_i(y)$.
- Remark that every Nash of some selection of $\mathcal{U}(x)$ will be an $2\varepsilon-$Nash of u.

Example 3

- Consider a strategic game with one player $i = 1$ (to simplify). Let M_1 is the set of mixed strategies of player 1, u the payoff function.
- Define
$$\mathcal{U}(\sigma) = u(\text{Support } \sigma).$$
- The preorder induced is $\sigma \lesssim_1 \sigma' \Leftrightarrow \sup_{x \in \text{Support}(\sigma)} u(x) \leq \inf_{x' \in \text{Support }(\sigma')} u(x')$.
- Too demanding (in general no equilibrium)
- In some sense, taking the expected payoff is a way to complete the preorder, and then an equilibrium generally exists.

Example 3

- Consider a strategic game with one player $i = 1$ (to simplify). Let M_1 is the set of mixed strategies of player 1, u the payoff function.
- Define
$$\mathcal{U}(\sigma) = u(\mathrm{Support}\ \sigma).$$
- The preorder induced is $\sigma \lesssim_1 \sigma' \Leftrightarrow \sup_{x \in \mathrm{Support}(\sigma)} u(x) \leq \inf_{x' \in \mathrm{Support}\ (\sigma')} u(x')$.
- Too demanding (in general no equilibrium)
- In some sense, taking the expected payoff is a way to complete the preorder, and then an equilibrium generally exists.

Example 3

- Consider a strategic game with one player $i = 1$ (to simplify). Let M_1 is the set of mixed strategies of player 1, u the payoff function.
- Define

$$\mathcal{U}(\sigma) = u(\text{Support } \sigma).$$

- The preorder induced is $\sigma \lesssim_1 \sigma' \Leftrightarrow \sup_{x \in \text{Support}(\sigma)} u(x) \leq \inf_{x' \in \text{Support }(\sigma')} u(x')$.
- Too demanding (in general no equilibrium)
- In some sense, taking the expected payoff is a way to complete the preorder, and then an equilibrium generally exists.

Example 3

- Consider a strategic game with one player $i = 1$ (to simplify). Let M_1 is the set of mixed strategies of player 1, u the payoff function.
- Define
$$\mathcal{U}(\sigma) = u(\text{Support } \sigma).$$
- The preorder induced is $\sigma \lesssim_1 \sigma' \Leftrightarrow \sup_{x \in \text{Support}(\sigma)} u(x) \leq \inf_{x' \in \text{Support }(\sigma')} u(x')$.
- Too demanding (in general no equilibrium)
- In some sense, taking the expected payoff is a way to complete the preorder, and then an equilibrium generally exists.

Example 3

- Consider a strategic game with one player $i = 1$ (to simplify). Let M_1 is the set of mixed strategies of player 1, u the payoff function.
- Define

$$\mathcal{U}(\sigma) = u(\text{Support } \sigma).$$

- The preorder induced is $\sigma \lesssim_1 \sigma' \Leftrightarrow \sup_{x \in \text{Support}(\sigma)} u(x) \leq \inf_{x' \in \text{Support } (\sigma')} u(x')$.
- Too demanding (in general no equilibrium)
- In some sense, taking the expected payoff is a way to complete the preorder, and then an equilibrium generally exists.

Example 3

- Consider a strategic game with one player $i = 1$ (to simplify). Let M_1 is the set of mixed strategies of player 1, u the payoff function.
- Define
$$\mathcal{U}(\sigma) = u(\text{Support } \sigma).$$
- The preorder induced is $\sigma \lesssim_1 \sigma' \Leftrightarrow$ $\sup_{x \in \text{Support}(\sigma)} u(x) \leq \inf_{x' \in \text{Support }(\sigma')} u(x')$.
- Too demanding (in general no equilibrium)
- In some sense, taking the expected payoff is a way to complete the preorder, and then an equilibrium generally exists.

1) d) Generalized games with endogenous sharing rule.

Last extension: the strategy space of each player depend on the strategies (see Arrow-Debreu 1954)

1) d) Generalized games with endogenous sharing rule

- A generalized game with endogenous sharing rule is $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$
- Here, player i has to play in $\mathcal{B}_i(x_{-i})$ given the strategies x_{-i} of the other players $-i$.
- Assume $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (e.g. a Kakutani-type mapping).
- When $\mathcal{U}$ is single-valued, we call the game "generalized game".

1) d) Generalized games with endogenous sharing rule

- A generalized game with endogenous sharing rule is $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$
- Here, player i has to play in $\mathcal{B}_i(x_{-i})$ given the strategies x_{-i} of the other players $-i$.
- Assume $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (e.g. a Kakutani-type mapping).
- When $\mathcal{U}$ is single-valued, we call the game "generalized game".

1) d) Generalized games with endogenous sharing rule

- A generalized game with endogenous sharing rule is $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$
- Here, player i has to play in $\mathcal{B}_i(x_{-i})$ given the strategies x_{-i} of the other players $-i$.
- Assume $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (e.g. a Kakutani-type mapping).
- When $\mathcal{U}$ is single-valued, we call the game "generalized game".

1) d) Generalized games with endogenous sharing rule

- A generalized game with endogenous sharing rule is $((X_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \mathcal{U})$
- Here, player i has to play in $\mathcal{B}_i(x_{-i})$ given the strategies x_{-i} of the other players $-i$.
- Assume $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (e.g. a Kakutani-type mapping).
- When $\mathcal{U}$ is single-valued, we call the game "generalized game".

1) d) Generalized games with endogenous sharing rule

- A generalized game with endogenous sharing rule is $((X_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \mathcal{U})$
- Here, player i has to play in $\mathcal{B}_i(x_{-i})$ given the strategies x_{-i} of the other players $-i$.
- Assume $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (e.g. a Kakutani-type mapping).
- When $\mathcal{U}$ is single-valued, we call the game "generalized game".

1) d) Generalized games with endogenous sharing rule

- A generalized game with endogenous sharing rule is $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$
- Here, player i has to play in $\mathcal{B}_i(x_{-i})$ given the strategies x_{-i} of the other players $-i$.
- Assume $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (e.g. a Kakutani-type mapping).
- When $\mathcal{U}$ is single-valued, we call the game "generalized game".

2) Solution concept; existence

We now pass to the solution concept, and its existence.

2) a) Solution of a Generalized games with endogenous sharing rule

Let $\mathcal{E} = ((X_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \mathcal{U})$ be a generalized game with endogenous sharing rule.

A solution of $\mathcal{E}$ is a couple (x, u), where:
(1) $u = (u_i)_{i\in N}$ is a selection of $\mathcal{U}$
(2) x is a generalized Nash equilibrium of $((X_i)_{i\in N}, (u_i)_{i\in N}, \mathcal{B})$:
(i) For every $i \in N$, $x_i \in \mathcal{B}_i(x_{-i})$.
(ii) For every $d_i \in \mathcal{B}_i(x_{-i})$, $u_i(d_i, x_{-i}) \leq u_i(x)$.

2) a) Solution of a Generalized games with endogenous sharing rule

- A similar notion exist, but (1) for games (2) in mixed strategies (Simon-Zame -Econometrica 1990.)
- It was designed principally to overcome the problem of discontinuity of payoffs.
- Existence of a solution in pure was an open question(Jackson et al. Econometrica 2002)
- Important because:
- 1) Pure strategies are generally more tractable in economic models.
- 2) Many equilibrium models in economics can be seen as discontinuous generalized games.

2) a) Solution of a Generalized games with endogenous sharing rule

- A similar notion exist, but (1) for games (2) in mixed strategies (Simon-Zame -Econometrica 1990.)
- It was designed principally to overcome the problem of discontinuity of payoffs.
- Existence of a solution in pure was an open question(Jackson et al. Econometrica 2002)
- Important because:
- 1) Pure strategies are generally more tractable in economic models.
- 2) Many equilibrium models in economics can be seen as discontinuous generalized games.

2) a) Solution of a Generalized games with endogenous sharing rule

- A similar notion exist, but (1) for games (2) in mixed strategies (Simon-Zame -Econometrica 1990.)
- It was designed principally to overcome the problem of discontinuity of payoffs.
- Existence of a solution in pure was an open question(Jackson et al. Econometrica 2002)
- Important because:
- 1) Pure strategies are generally more tractable in economic models.
- 2) Many equilibrium models in economics can be seen as discontinuous generalized games.

2) a) Solution of a Generalized games with endogenous sharing rule

- A similar notion exist, but (1) for games (2) in mixed strategies (Simon-Zame -Econometrica 1990.)
- It was designed principally to overcome the problem of discontinuity of payoffs.
- Existence of a solution in pure was an open question(Jackson et al. Econometrica 2002)
- Important because:
- 1) Pure strategies are generally more tractable in economic models.
- 2) Many equilibrium models in economics can be seen as discontinuous generalized games.

2) a) Solution of a Generalized games with endogenous sharing rule

- A similar notion exist, but (1) for games (2) in mixed strategies (Simon-Zame -Econometrica 1990.)
- It was designed principally to overcome the problem of discontinuity of payoffs.
- Existence of a solution in pure was an open question(Jackson et al. Econometrica 2002)
- Important because:
- 1) Pure strategies are generally more tractable in economic models.
- 2) Many equilibrium models in economics can be seen as discontinuous generalized games.

2) a) Solution of a Generalized games with endogenous sharing rule

- A similar notion exist, but (1) for games (2) in mixed strategies (Simon-Zame -Econometrica 1990.)
- It was designed principally to overcome the problem of discontinuity of payoffs.
- Existence of a solution in pure was an open question(Jackson et al. Econometrica 2002)
- Important because:
- 1) Pure strategies are generally more tractable in economic models.
- 2) Many equilibrium models in economics can be seen as discontinuous generalized games.

2) b) Example of generalized game: Exchange economy

- n consumers and m commodities.
- Consumer i's consumption set is X_i, initial endowment e_i.
- Utility function $u_i : X_i \to \mathbf{R}+$.
- Walrasian equilibrium: $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$ is generalized Nash equilibrium of $N+1$-games, where:
- Each player $i = 1, ..., N$ solve

$$\sup_{x_i \in X_i} \tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } p.(x_i - e_i) \leq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- Fictive player $N+1$ solves

$$\sup_{p \in \Delta(\mathbf{R}^m_+)} p.(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} e_i).$$

2) b) Example of generalized game: Exchange economy

- n consumers and m commodities.
- Consumer i's consumption set is X_i, initial endowment e_i.
- Utility function $u_i : X_i \to \mathbf{R}+$.
- Walrasian equilibrium: $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$ is generalized Nash equilibrium of $N+1$-games, where:
- Each player $i = 1, ..., N$ solve

$$\sup_{x_i \in X_i} \tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } p.(x_i - e_i) \leq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- Fictive player $N+1$ solves

$$\sup_{p \in \Delta(\mathbf{R}^m_+)} p.(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} e_i).$$

2) b) Example of generalized game: Exchange economy

- n consumers and m commodities.
- Consumer i's consumption set is X_i, initial endowment e_i.
- Utility function $u_i : X_i \to \mathbf{R}+$.
- Walrasian equilibrium: $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}_+^m)$ is generalized Nash equilibrium of $N+1$-games, where:
- Each player $i = 1, ..., N$ solve

$$\sup_{x_i \in X_i} \tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } p.(x_i - e_i) \leq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- Fictive player $N+1$ solves

$$\sup_{p \in \Delta(\mathbf{R}_+^m)} p.(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} e_i).$$

2) b) Example of generalized game: Exchange economy

- n consumers and m commodities.
- Consumer i's consumption set is X_i, initial endowment e_i.
- Utility function $u_i : X_i \to \mathbf{R}+$.
- Walrasian equilibrium: $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$ is generalized Nash equilibrium of $N+1$-games, where:
- Each player $i = 1, ..., N$ solve

$$\sup_{x_i \in X_i} \tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } p.(x_i - e_i) \leq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- Fictive player $N+1$ solves

$$\sup_{p \in \Delta(\mathbf{R}^m_+)} p.(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} e_i).$$

2) b) Example of generalized game: Exchange economy

- n consumers and m commodities.
- Consumer i's consumption set is X_i, initial endowment e_i.
- Utility function $u_i : X_i \to \mathbf{R}+$.
- Walrasian equilibrium: $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$ is generalized Nash equilibrium of $N+1$-games, where:
- Each player $i = 1, ..., N$ solve

$$\sup_{x_i \in X_i} \tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } p.(x_i - e_i) \leq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- Fictive player $N+1$ solves

$$\sup_{p \in \Delta(\mathbf{R}^m_+)} p.(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} e_i).$$

2) b) Example of generalized game: Exchange economy

- n consumers and m commodities.
- Consumer i's consumption set is X_i, initial endowment e_i.
- Utility function $u_i : X_i \to \mathbf{R}+$.
- Walrasian equilibrium: $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}_+^m)$ is generalized Nash equilibrium of $N+1$-games, where:
- Each player $i = 1, ..., N$ solve

$$\sup_{x_i \in X_i} \tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } p.(x_i - e_i) \leq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- Fictive player $N+1$ solves

$$\sup_{p \in \Delta(\mathbf{R}_+^m)} p.(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} e_i).$$

2) c) Existence of a solution

Any generalized game with endogenous sharing rules $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$ satisfying $(A1)$ to $(A5)$ below admits a solution.

- $A1$: X is a convex and compact subset of a Hausdorff and locally convex topological vector space;
- $A2$: $\mathcal{U}$ is bounded;
- $A3$: The graph of the economy $\Gamma := \{(x, v) : v \in \mathcal{U}(x)$ and $\forall i \in N \;\; x_i \in \mathcal{B}_i(x_{-i})\}$ is closed;
- $A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i \in N}$, such that each u_i is quasiconcave in player i's strategy.
- $A5$ The B_i are Kakutani's multivalued functions.

2) c) Existence of a solution

Any generalized game with endogenous sharing rules $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$ satisfying $(A1)$ to $(A5)$ below admits a solution.

- $A1$: X is a convex and compact subset of a Hausdorff and locally convex topological vector space;
- $A2$: $\mathcal{U}$ is bounded;
- $A3$: The graph of the economy $\Gamma := \{(x, v) : v \in \mathcal{U}(x)$ and $\forall i \in N \;\; x_i \in \mathcal{B}_i(x_{-i})\}$ is closed;
- $A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i \in N}$, such that each u_i is quasiconcave in player i's strategy.
- $A5$ The B_i are Kakutani's multivalued functions.

2) c) Existence of a solution

Any generalized game with endogenous sharing rules $((X_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \mathcal{U})$ satisfying $(A1)$ to $(A5)$ below admits a solution.

- $A1$: X is a convex and compact subset of a Hausdorff and locally convex topological vector space;
- $A2$: $\mathcal{U}$ is bounded;
- $A3$: The graph of the economy $\Gamma := \{(x, v) : v \in \mathcal{U}(x)$ and $\forall i \in N \;\; x_i \in \mathcal{B}_i(x_{-i})\}$ is closed;
- $A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i\in N}$, such that each u_i is quasiconcave in player i's strategy.
- $A5$ The B_i are Kakutani's multivalued functions.

2) c) Existence of a solution

Any generalized game with endogenous sharing rules $((X_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \mathcal{U})$ satisfying $(A1)$ to $(A5)$ below admits a solution.

- $A1$: X is a convex and compact subset of a Hausdorff and locally convex topological vector space;
- $A2$: $\mathcal{U}$ is bounded;
- $A3$: The graph of the economy $\Gamma := \{(x, v) : v \in \mathcal{U}(x)$ and $\forall i \in N \;\; x_i \in \mathcal{B}_i(x_{-i})\}$ is closed;
- $A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i\in N}$, such that each u_i is quasiconcave in player i's strategy.
- $A5$ The B_i are Kakutani's multivalued functions.

2) c) Existence of a solution

Any generalized game with endogenous sharing rules $((X_i)_{i \in N}, (\mathcal{B}_i)_{i \in N}, \mathcal{U})$ satisfying $(A1)$ to $(A5)$ below admits a solution.

- $A1$: X is a convex and compact subset of a Hausdorff and locally convex topological vector space;
- $A2$: $\mathcal{U}$ is bounded;
- $A3$: The graph of the economy $\Gamma := \{(x, v) : v \in \mathcal{U}(x)$ and $\forall i \in N \;\; x_i \in \mathcal{B}_i(x_{-i})\}$ is closed;
- $A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i \in N}$, such that each u_i is quasiconcave in player i's strategy.
- $A5$ The B_i are Kakutani's multivalued functions.

3) Proof.

First idea: find good behaved selections of $\mathcal{U}$ (continuous, quasiconcave) and apply general existence result for games or generalized games.

But $\mathcal{U}$ has no such selection in general.

3) Proof.

First idea: find good behaved selections of $\mathcal{U}$ (continuous, quasiconcave) and apply general existence result for games or generalized games.

But $\mathcal{U}$ has no such selection in general.

3) Proof.

Second idea (the good one)

- Step 1: Consider the quasiconcave selection u; associate to the generalized game define by u a discontinuous game.
- Step 2: For this discontinuous game, existence of "weak" generalized Nash equilibrium $\bar{x}$, for which players play optimally up to infinitesimal modifications of their strategies.
- Step 3: Modify "slightly" the game at every $(d_i, \bar{x}_{-i})$, $d_i \in X_i$, $i \in I$, so that $\bar{x}$ is a generalized Nash of the new generalized game associated.

3) Proof.

Second idea (the good one)

- Step 1: Consider the quasiconcave selection u; associate to the generalized game define by u a discontinuous game.
- Step 2: For this discontinuous game, existence of "weak" generalized Nash equilibrium $\bar{x}$, for which players play optimally up to infinitesimal modifications of their strategies.
- Step 3: Modify "slightly" the game at every $(d_i, \bar{x}_{-i})$, $d_i \in X_i$, $i \in I$, so that $\bar{x}$ is a generalized Nash of the new generalized game associated.

3) Proof.

Second idea (the good one)

- Step 1: Consider the quasiconcave selection u; associate to the generalized game define by u a discontinuous game.
- Step 2: For this discontinuous game, existence of "weak" generalized Nash equilibrium $\bar{x}$, for which players play optimally up to infinitesimal modifications of their strategies.
- Step 3: Modify "slightly" the game at every $(d_i, \bar{x}_{-i})$, $d_i \in X_i$, $i \in I$, so that $\bar{x}$ is a generalized Nash of the new generalized game associated.

3) Proof.

Second idea (the good one)

- Step 1: Consider the quasiconcave selection u; associate to the generalized game define by u a discontinuous game.
- Step 2: For this discontinuous game, existence of "weak" generalized Nash equilibrium $\bar{x}$, for which players play optimally up to infinitesimal modifications of their strategies.
- Step 3: Modify "slightly" the game at every $(d_i, \bar{x}_{-i})$, $d_i \in X_i$, $i \in I$, so that $\bar{x}$ is a generalized Nash of the new generalized game associated.

3) Proof. Step 1: Associate to generalized game a discontinuous game.

Following an idea of Reny (Econometrica 1999), we associate to $(G = ((X_i)_{i\in N}, u, (\mathcal{B}_i)_{i\in N})$ a strategic game G' as follows:

- Let $\Lambda \in \mathbb{R}$ such that $u_i(x) \geq \Lambda + 1$ for every $i \in N$ and every profile $x \in X$.
- The game G' has N players.
- In G', strategy set of player i is X_i, and his payoff is

$$v_i(x) = \begin{cases} u_i(x) & \text{if } x_i \in \mathcal{B}_i(x_{-i}), \\ \Lambda & \text{otherwise.} \end{cases}$$

G' is a quasiconcave game.

3) Proof. Step 1: Associate to generalized game a discontinuous game.

Following an idea of Reny (Econometrica 1999), we associate to $(G = ((X_i)_{i \in N}, u, (\mathcal{B}_i)_{i \in N})$ a strategic game G' as follows:

- Let $\Lambda \in \mathbb{R}$ such that $u_i(x) \geq \Lambda + 1$ for every $i \in N$ and every profile $x \in X$.
- The game G' has N players.
- In G', strategy set of player i is X_i, and his payoff is

$$v_i(x) = \begin{cases} u_i(x) & \text{if } x_i \in \mathcal{B}_i(x_{-i}), \\ \Lambda & \text{otherwise.} \end{cases}$$

  G' is a quasiconcave game.

3) Proof. Step 1: Associate to generalized game a discontinuous game.

Following an idea of Reny (Econometrica 1999), we associate to $(G = ((X_i)_{i \in N}, u, (\mathcal{B}_i)_{i \in N})$ a strategic game G' as follows:

- Let $\Lambda \in \mathbb{R}$ such that $u_i(x) \geq \Lambda + 1$ for every $i \in N$ and every profile $x \in X$.
- The game G' has N players.
- In G', strategy set of player i is X_i, and his payoff is

$$v_i(x) = \begin{cases} u_i(x) & \text{if } x_i \in \mathcal{B}_i(x_{-i}), \\ \Lambda & \text{otherwise.} \end{cases}$$

G' is a quasiconcave game.

3) Proof. Step 2: Weak notion of generalized Nash equilibrium.

Theorem (Bich-Laraki): Every quasiconcave game $(G = ((X_i)_{i\in N}, v)$ admits a weak equilibrium:
it is $(x, \alpha) \in \overline{(x, v(x)) : x \in X\}}$ such that for every $i \in N$,

$$\sup_{d_i \in X_i} \underline{\underline{v_i}}(d_i, x_{-i}) \leq \alpha_i. \tag{1}$$

$$\underline{\underline{v_i}}(x) := \sup_{U\in\mathcal{V}(x_{-i})} \sup_{d_i\in W_U(x)} \inf_{x'_{-i}\in U,\ x'_i\in d_i(x'_{-i})} v_i(x').$$

$\mathcal{V}(x_{-i})$ is the set of open neighborhood of x_{-i}
$W_U(x_i, x_{-i})$ is set of Kakutani-type multivalued mappings $d_i(.)$ from U to X_i such that $x_i \in d_i(x_{-i})$.

3) Proof. Step 2: Weak notion of generalized Nash equilibrium.

Theorem (Bich-Laraki): Every quasiconcave game $(G = ((X_i)_{i \in N}, v)$ admits a weak equilibrium:
it is $(x, \alpha) \in \overline{(x, v(x)) : x \in X\}}$ such that for every $i \in N$,

$$\sup_{d_i \in X_i} \underline{\underline{v_i}}(d_i, x_{-i}) \leq \alpha_i. \tag{1}$$

$$\underline{\underline{v_i}}(x) := \sup_{U \in \mathcal{V}(x_{-i})} \sup_{d_i \in W_U(x)} \inf_{x'_{-i} \in U,\ x'_i \in d_i(x'_{-i})} v_i(x').$$

$\mathcal{V}(x_{-i})$ is the set of open neighborhood of x_{-i}
$W_U(x_i, x_{-i})$ is set of Kakutani-type multivalued mappings $d_i(.)$ from U to X_i such that $x_i \in d_i(x_{-i})$.

3) Proof. Step 2: Weak notion of generalized Nash equilibrium.

Theorem (Bich-Laraki): Every quasiconcave game $(G = ((X_i)_{i \in N}, v)$ admits a weak equilibrium:
it is $(x, \alpha) \in \overline{(x, v(x)) : x \in X\}}$ such that for every $i \in N$,

$$\sup_{d_i \in X_i} \underline{\underline{v_i}}(d_i, x_{-i}) \leq \alpha_i. \quad (1)$$

$$\underline{\underline{v_i}}(x) := \sup_{U \in \mathcal{V}(x_{-i})} \sup_{d_i \in W_U(x)} \inf_{x'_{-i} \in U,\ x'_i \in d_i(x'_{-i})} v_i(x').$$

$\mathcal{V}(x_{-i})$ is the set of open neighborhood of x_{-i}
$W_U(x_i, x_{-i})$ is set of Kakutani-type multivalued mappings $d_i(.)$ from U to X_i such that $x_i \in d_i(x_{-i})$.

3) Proof. Step 2: Weak notion of generalized Nash equilibrium.

Theorem (Bich-Laraki): Every quasiconcave game $(G = ((X_i)_{i\in N}, v)$ admits a weak equilibrium:
it is $(x, \alpha) \in \overline{(x, v(x)) : x \in X\}}$ such that for every $i \in N$,

$$\sup_{d_i \in X_i} \underline{\underline{v_i}}(d_i, x_{-i}) \leq \alpha_i. \qquad (1)$$

$$\underline{\underline{v_i}}(x) := \sup_{U\in\mathcal{V}(x_{-i})} \sup_{d_i\in W_U(x)} \inf_{x'_{-i}\in U,\ x'_i\in d_i(x'_{-i})} v_i(x').$$

$\mathcal{V}(x_{-i})$ is the set of open neighborhood of x_{-i}
$W_U(x_i, x_{-i})$ is set of Kakutani-type multivalued mappings $d_i(.)$ from U to X_i such that $x_i \in d_i(x_{-i})$.

3) Proof. Step 1: Weak notion of generalized Nash equilibrium.

The proof is by contradiction:

- otherwise, the game is generalized-secure, one can apply Barelli-Soza (Econometrica 2014) to get a Nash.
- But a Nash is a weak equilibrium, thus contradiction.

3) Proof. Step 1: Weak notion of generalized Nash equilibrium.

The proof is by contradiction:

- otherwise, the game is generalized-secure, one can apply Barelli-Soza (Econometrica 2014) to get a Nash.
- But a Nash is a weak equilibrium, thus contradiction.

3) Proof. Step 1: Weak notion of generalized Nash equilibrium.

The proof is by contradiction:

- otherwise, the game is generalized-secure, one can apply Barelli-Soza (Econometrica 2014) to get a Nash.
- But a Nash is a weak equilibrium, thus contradiction.

3) Proof. Step 3: Existence of a solution.

Let (x, α) weak equilibrium of $G' = ((X_i)_{i \in N}, (v_i)_{i \in N})$

Define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} \alpha & \text{if } y = x, \\ \text{any limit point of } v(x^n)_{n \in \mathbf{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x^n)_{n \in \mathbf{N}} \in \underline{\underline{\mathcal{S}_i}}(d_i, x_{-i}), \\ q(y) = v(y) & \text{otherwise.} \end{cases}$$

where $\underline{\underline{\mathcal{S}_i}}(x)$= space of sequences $(x^n)_{n \in \mathbf{N}}$ of X converging to x such that $\lim_{n \to +\infty} v_i(x^n) = \underline{\underline{v_i}}(x)$.

One can prove easily that q provides a solution, i.e. a selection of $\mathcal{U}$, with x generalized Nash of q.

3) Proof. Step 3: Existence of a solution.

Let (x, α) weak equilibrium of $G' = ((X_i)_{i\in N}, (v_i)_{i\in N})$
Define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} \alpha & \text{if } y = x, \\ \text{any limit point of } v(x^n)_{n\in\mathbf{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x^n)_{n\in\mathbf{N}} \in \underline{\underline{\mathcal{S}_i}}(d_i, x_{-i}), \\ q(y) = v(y) & \text{otherwise.} \end{cases}$$

where $\underline{\underline{\mathcal{S}_i}}(x)$= space of sequences $(x^n)_{n\in\mathbf{N}}$ of X converging to x such that $\lim_{n\to+\infty} v_i(x^n) = \underline{\underline{v_i}}(x)$.

One can prove easily that q provides a solution, i.e. a selection of $\mathcal{U}$, with x generalized Nash of q.

3) Proof. Step 3: Existence of a solution.

Let (x, α) weak equilibrium of $G' = ((X_i)_{i\in N}, (v_i)_{i\in N})$
Define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} \alpha & \text{if } y = x, \\ \text{any limit point of } v(x^n)_{n\in\mathbf{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N,\ d_i \neq x_i,\ (x^n)_{n\in\mathbf{N}} \in \underline{\underline{\mathcal{S}_i}}(d_i, x_{-i}), \\ q(y) = v(y) & \text{otherwise.} \end{cases}$$

where $\underline{\underline{\mathcal{S}_i}}(x)$= space of sequences $(x^n)_{n\in\mathbf{N}}$ of X converging to x such that $\lim_{n\to+\infty} v_i(x^n) = \underline{\underline{v_i}}(x)$.

One can prove easily that q provides a solution, i.e. a selection of $\mathcal{U}$, with x generalized Nash of q.

3) Proof. Step 3: Existence of a solution.

Let (x, α) weak equilibrium of $G' = ((X_i)_{i\in N}, (v_i)_{i\in N})$
Define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} \alpha & \text{if } y = x, \\ \text{any limit point of } v(x^n)_{n\in\mathbf{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N,\, d_i \neq x_i,\, (x^n)_{n\in\mathbf{N}} \in \underline{\underline{\mathcal{S}_i}}(d_i, x_{-i}), \\ q(y) = v(y) & \text{otherwise.} \end{cases}$$

where $\underline{\underline{\mathcal{S}_i}}(x)$= space of sequences $(x^n)_{n\in\mathbf{N}}$ of X converging to x such that $\lim_{n\to+\infty} v_i(x^n) = \underline{\underline{v_i}}(x)$.

One can prove easily that q provides a solution, i.e. a selection of $\mathcal{U}$, with x generalized Nash of q.

Interpretation with incomplete preferences.

Let $\mathcal{E} = ((X_i)_{i \in N}, (\lesssim_i)_{i \in N})$ be an ordered game, i.e. preferences $\lesssim_i$ are preorders.

A Nash equilibrium of $\mathcal{E}$ is a profile $x \in \Pi_{i \in N} X_i$ such that there is no player $i \in N$ and no $y_i \in X_i$ with $x \precnsim_i (y_i, x_{-i})$.

Consider again the preorders $x \lesssim_i y \Leftrightarrow$ for every selection $u = (u_1, ..., u_N)$ of $\mathcal{U}$, $u_i(x) \leq u_i(y)$.

Under assumption (A1)-(A4) on $\mathcal{U}$, $\mathcal{E}$ admits a completion v such that $((X_i)_{i \in N}, v)$ has a Nash equilibrium.

Interpretation with incomplete preferences.

Let $\mathcal{E} = ((X_i)_{i\in N}, (\lesssim_i)_{i\in N})$ be an ordered game, i.e. preferences $\lesssim_i$ are preorders.

A Nash equilibrium of $\mathcal{E}$ is a profile $x \in \Pi_{i\in N} X_i$ such that there is no player $i \in N$ and no $y_i \in X_i$ with $x \lnsim_i (y_i, x_{-i})$.

Consider again the preorders $x \lesssim_i y \Leftrightarrow$ for every selection $u = (u_1, ..., u_N)$ of $\mathcal{U}$, $u_i(x) \leq u_i(y)$.

Under assumption (A1)-(A4) on $\mathcal{U}$, $\mathcal{E}$ admits a completion v such that $((X_i)_{i\in N}, v)$ has a Nash equilibrium.

Interpretation with incomplete preferences.

Let $\mathcal{E} = ((X_i)_{i \in N}, (\lesssim_i)_{i \in N})$ be an ordered game, i.e. preferences $\lesssim_i$ are preorders.

A Nash equilibrium of $\mathcal{E}$ is a profile $x \in \Pi_{i \in N} X_i$ such that there is no player $i \in N$ and no $y_i \in X_i$ with $x \lnsim_i (y_i, x_{-i})$.

Consider again the preorders $x \lesssim_i y \Leftrightarrow$ for every selection $u = (u_1, ..., u_N)$ of $\mathcal{U}$, $u_i(x) \leq u_i(y)$.

Under assumption (A1)-(A4) on $\mathcal{U}$, $\mathcal{E}$ admits a completion v such that $((X_i)_{i \in N}, v)$ has a Nash equilibrium.

Interpretation with incomplete preferences.

Let $\mathcal{E} = ((X_i)_{i \in N}, (\lesssim_i)_{i \in N})$ be an ordered game, i.e. preferences $\lesssim_i$ are preorders.

A Nash equilibrium of $\mathcal{E}$ is a profile $x \in \Pi_{i \in N} X_i$ such that there is no player $i \in N$ and no $y_i \in X_i$ with $x \lnsim_i (y_i, x_{-i})$.

Consider again the preorders $x \lesssim_i y \Leftrightarrow$ for every selection $u = (u_1, ..., u_N)$ of $\mathcal{U}$, $u_i(x) \leq u_i(y)$.

Under assumption (A1)-(A4) on $\mathcal{U}$, $\mathcal{E}$ admits a completion v such that $((X_i)_{i \in N}, v)$ has a Nash equilibrium.

Thank you.