Zero-sum Stopping Games with Asymmetric Information.

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Plan

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Stopping Games with Asymmetric information

- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- Markov process $(X_t, Y_t)_{t\geq 0}$ with values in $K \times L$.
- $m \in \Delta(K \times L)$: initial distribution.
- $r > 0$: discount factor.
- Two bounded measurable functions $f, h : K \times L \to \mathbb{R}$.

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- Two bounded measurable functions $f, h : K \times L \to \mathbb{R}$.
- Player 1 chooses an $\mathcal{F}^X$-(randomized) stopping time μ.
  Player 2 chooses an $\mathcal{F}^Y$-(randomized) stopping time τ.
- The expected payoff when players choose (μ, τ) is :

$$J_m(\mu, \tau) = \mathbb{E}_m[e^{-r\tau} f(X_\tau, Y_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \tau}].$$

- Player 1 is maximizing, and Player 2 is minimizing.

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- Player 1 is maximizing, and Player 2 is minimizing.

Define the lower and upper values as :

$$V^-(m) := \sup_\mu \inf_\tau J_m(\mu, \tau), \;\; V^+(m) := \inf_\tau \sup_\mu J_m(\mu, \tau).$$

Stopping Games with Asymmetric information

Questions :

- Existence of the value (allowing or not for randomized stopping times) ?
- Existence/characterization of optimal strategies ?

Some references for the symmetric information case $X \equiv Y$

E. Dynkin : Game variant of a problem of optimal stopping, *Soviet. Math. Dokl.* 10 : 270-274, 1967.

- Analytical Approach

  Friedmann (1973), Bensoussan, Friedman (1974), Stettner (1982),...
- Stochastic Approach

  Bismut (1977), Alario-Nazaret, Lepeltier, Marchal (1982), Lepeltier, Maingueneau (1984), Ekström, Peskir (2008), Kobylanski, Quenez, de Campagnolle (2012), ...
  Touzi, Vieille (2000), Laraki, Solan (2005)
- BSDE Approach - Switching

  Cvitaniċ, Karatzas (1996), Hamadène, Lepeltier (2000), Hamadène, Hassani (2005), Buckdahn, Li (2008), ...
  Hu, Tang (2008), El Asri, Hamadène (2009), Hamadène, Zhang (2010), ...

Classical results in the symmetric information case $X \equiv Y$.

Theorem[Lepeltier-Maingueneau(1984), Ekström-Peskir(2008)]

If $h \leq f$ are continuous and X is right-continuous, then the value exists.
If moreover X is quasi-left-continuous, there exist optimal hitting time strategies.

The hitting time of $A \subset K$ is the first time X enters in A.

Remark 1 : When X is a diffusion process, the value is the unique viscosity solution of some non-linear PDE (double obstacle).

Remark 2 : For more general payoff functions, the value may not exist. However, it always exist if we allow the players to use randomized stopping times (Laraki-Solan 2005).

Existing works in the asymmetric information case.

- C.Grün(2013) : Dynkin Games with incomplete information.

  Case $h \leq f$, $X_t = (\xi, B_t)$ and $Y_t = B_t$ where B is a Brownian motion and ξ is a random variable taking finitely may values and independent of B.

  Existence of the value with randomized stopping times.

  Characterization as the unique viscosity solution of a PDE and as the solution of a stochastic optimization problem over beliefs.
- Lempa-Matomaki(2010) : asymmetric information on the discount rate.

In this work : We extend the results of C. Grün to a different framework without Brownian motion but where the parameters on which there is incomplete information are evolving.

Plan

Model and assumptions.

- K, L : non-empty finite sets.
- $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent continuous-time Markov chains with values in K, L.
- X :initial distribution $p \in \Delta(K)$, infinitesimal generator $R = (R_{k,k'})_{k,k'\in K}$.
- Y : initial distribution $q \in \Delta(L)$, infinitesimal generator $Q = (Q_{\ell,\ell'})_{\ell,\ell'\in L}$.
- $\mathbb{P}_{p,q}$: joint law of (X, Y) on the canonical space

$$\Omega_X \times \Omega_Y := \mathbb{D}([0,\infty), K) \times \mathbb{D}([0,\infty), L).$$

$$\mathbb{P}_{p,q}[X_t = k] = (e^{t\,{}^{\top}R} p)_k.$$

$$\mathbb{P}_{p,q}[Y_t = \ell] = (e^{t\,{}^{\top}Q} q)_\ell.$$

The game

Stopping game, with discount $r > 0$

(i) payoff when P1 stops first : $h : K \times L \to \mathbb{R}$,

(ii) payoff when P2 stops first : $f : K \times L \to \mathbb{R}$

such that $h \leq f$ and a payoff functional

$$e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \tau} + e^{-r\tau} f(X_\tau, Y_\tau) \mathbb{1}_{\tau < \mu}.$$

P1 chooses a randomized stopping time μ of $\mathcal{F}^X$ in order to maximize the expected payoff, P2 chooses a randomized stopping time τ of $\mathcal{F}^Y$ in order to minimize it.

Information Asymmetry :

- P1 only observes X,
- P2 only observes Y,
- the joint distribution is common knowledge.

Definition : Randomized stopping times

Definition

A randomized stopping time of $\mathcal{F}^X$ is a measurable function $\mu : [0,1] \times \Omega_X \to [0,+\infty]$, such that for all $u \in [0,1]$

$$\mu_u(.) := \mu(u,.)$$

is an $\mathcal{F}^X$ stopping time.
A randomized stopping time $\tau(v,.)$ of $\mathcal{F}^Y$ is defined similarly.

For all $(p,q) \in \Delta(K) \times \Delta(L)$, $\mu \in \mathcal{T}_r^X, \tau \in \mathcal{T}_r^Y$ we set

$$J(p,q,\mu,\nu) = \mathbb{E}_{p,q}\left[\int_0^1 \int_0^1 (e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \nu} + e^{-r\tau} f(X_\tau, Y_\tau) \mathbb{1}_{\tau < \mu})\, dudv \right].$$

The upper and lower value functions

For $(p, q) \in \Delta(K) \times \Delta(L)$, we define the and the lower value function by

$$V^{-}(p,q) = \sup_{\mu \in T_r^X} \inf_{\tau \in \mathcal{T}_r^Y} J(p,q,\mu,\nu)$$

and the upper value function by

$$V^{+}(p,q) = \inf_{\tau \in \mathcal{T}_r^Y} \sup_{\mu \in T_r^X} J(p,q,\mu,\nu).$$

Note : By definition $V^{-}(p) \leq V^{+}(p)$.

Notations :

- If $G : \Delta(K) \times \Delta(L) \to \mathbb{R}$ is concave-convex,
  $Ext(G(\cdot, q))$: extreme points of the hypograph of $G(\cdot, q)$.
  $Ext(G(p, \cdot))$: extreme points of the epigraph of $G(p, \cdot)$.
- $\vec{D}_1 G(p, q; \xi)$, $\vec{D}_2 G(p, q; \zeta)$: directional derivatives of G at (p, q) with respect to the first and second variables in the directions ξ and ζ.
- Let f, h be extended bi-linearly on the set $\Delta(K) \times \Delta(L)$, i.e.

$$f(p, q) := \sum_{(k,\ell) \in K \times L} p_k q_\ell f(k, \ell),$$

$$h(p, q) := \sum_{(k,\ell) \in K \times L} p_k q_\ell h(k, \ell).$$

Existence and Characterization of the value.

Theorem

For all $(p,q) \in \Delta(K) \times \Delta(L)$*, the game has a value*

$$V(p,q) := V^+(p,q) = V^-(p,q),$$

V *is the unique concave-convex Lipschitz function on* $\Delta(K) \times \Delta(L)$ *such that* $h \le V \le f$ *and :*

Subsolution *:* $\forall q \in \Delta(L)$, $\forall p \in Ext(V(.,q))$,

$$V(p,q) > h(p,q) \Rightarrow rV(p,q) - \vec{D}_1 V(p,q;{}^{\top}Rp) - \vec{D}_2 V(p,q;{}^{\top}Qq) \le 0, \quad (1)$$

Supersolution *:* $\forall p \in \Delta(K)$, $\forall q \in Ext(V(p,.))$,

$$V(p,q) < f(p,q) \Rightarrow rV(p,q) - \vec{D}_1 V(p,q;{}^{\top}Rp) - \vec{D}_2 V(p,q;{}^{\top}Qq) \ge 0, \quad (2)$$

(quite similar to Mertens-Zamir(71) and Laraki(01),Rosenberg-Sorin(01))

Existence and Characterization of the value.

Subsolution : $\forall q \in \Delta(L)$, $\forall p \in Ext(V(.,q))$,

$$V(p,q) > h(p,q) \Rightarrow rV(p,q) - \vec{D}_1 V(p,q; {}^{\top}Rp) - \vec{D}_2 V(p,q; {}^{\top}Qq) \leq 0, \quad (1)$$

Comments on (1) **:**

- p is an extreme point if $V(.,q)$ is strictly concave at p (or p is an extreme point of $\Delta(K)$).
- In this case, (1) is a standard subsolution property for an obstacle first order PDE, in a pointwise sense because directional derivatives are always well-defined.
- This obstacle PDE can be associated to a stopping game with a deterministic dynamic on $\Delta(K) \times \Delta(L)$ given by the marginal distribution of (X_t, Y_t), i.e. $(e^{t\,{}^{\top}R}p, e^{t\,{}^{\top}Q}q)$ for some initial values $(p,q) \in \Delta(K) \times \Delta(L)$. This corresponds exactly to the characterization of the value of the game where both players do not observe (X_t, Y_t).

Existence and Characterization of the value.

Remark on randomized stopping times :

- The value may not exist if we do not allow for randomized stopping times, and the lower value can be non-concave with respect to p (resp. the upper value non-convex with respect to q).
- Allowing for randomized stopping times has two effects : convexity and compactness.
- Compactness is important to guarantee existence of optimal stopping times.
- Non-existence of the value without randomization is due to the lack of convexity in this model and to the fact that we consider Markov processes with finite state space. It is not clear whether the same phenomenon will occur for diffusions.

Main idea of the proof : Duality

(Based on ideas of De Meyer(96), Cardaliaguet(07) and Cardaliaguet-Rainer(09))

General scheme : We prove that V^- is supersolution of the initial equation. By symmetry, V^+ is subsolution, and we conclude that $V^+ \leq V^-$ using a comparison result (similar to Mertens-Zamir(71)).

- We do not how to prove directly the subsolution property for V^-.
  Intuition : P1 cannot compute posteriors without knowing the strategy of P2, and therefore cannot play optimally in a continuation game at time $t > 0$.

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  Intuition : P1 cannot compute posteriors without knowing the strategy of P2, and therefore cannot play optimally in a continuation game at time $t > 0$.
- Instead, we look at the convex conjugate

$$V_*^-(p,y) := \sup_{q \in \Delta(L)} \langle q, y \rangle - V^-(p,q).$$

  V_*^- is globally convex and we prove that V_*^- satisfies a sub-DPP and is therefore a subsolution of the dual equation :

  For all $(p,y) \in \Delta(K) \times \mathbb{R}^L$:

$$V_*^-(p,y) > f_*(p,y) \Rightarrow rV_*^-(p,y) - \vec{D}V_*^-(p,y; {}^{\top}Rp, (rI - Q)y) \leq 0,$$

  where f_*, h_* are the conjugates of f, h respectively.

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  Intuition : P1 cannot compute posteriors without knowing the strategy of P2, and therefore cannot play optimally in a continuation game at time $t > 0$.
- Instead, we look at the convex conjugate

  $$V^-_*(p,y) := \sup_{q \in \Delta(L)} \langle q, y \rangle - V^-(p,q).$$

  V^-_* is globally convex and we prove that V^-_* satisfies a sub-DPP and is therefore a subsolution of the dual equation :

  For all $(p,y) \in \Delta(K) \times \mathbb{R}^L$:

  $$V^-_*(p,y) > f_*(p,y) \Rightarrow rV^-_*(p,y) - \vec{D}V^-_*(p,y; {}^\top Rp, (rI - Q)y) \leq 0,$$

  where f_*, h_* are the conjugates of f, h respectively.
- We deduce that V^- is subsolution of the initial equation using arguments from convex analysis.

The conjugate value function V_*.

- The conjugate function

$$V_*(p,y) := \sup_{q \in \Delta(K)} \langle y, q \rangle - V(p,q),$$

is globally convex, such that $h_* \geq V_* \geq f_*$ and V_* is a subsolution of the dual equation :

For all $(p,y) \in \Delta(K) \times \mathbb{R}^L$,

$$V_*(p,y) > f_*(p,y) \Rightarrow rV_*(p,y) - \vec{D}V_*(p,y; {}^{\top}Rp, (rI - Q)y) \leq 0.$$

- V_* may be interpreted as the value of a dual game with incomplete information on one side (for player 1).
- Using the dual variables, we can provide a certificate of optimality (verification) for stopping times of P1 (and symmetrically for P2).

(augmented) Beliefs processes.

Assume that Player 1 chooses a randomized stopping time μ.

- Define the belief process $(\pi_t)_{t\geq 0}$ of Player 2 taking values in $\Delta(K)$ by :

$$\forall k \in K,\ (\pi_t)_k = \mathbb{P}_{p,q}(X_t = k | \mathbb{1}_{\mu > t}).$$

- $\mathbb{E}_{p,q}[\pi_0] = p$.
- Using the Markov property, we check easily that for $0 \leq s \leq t$:

$$\mathbb{E}_{p,q}[\pi_t | \mathcal{F}_s^\pi] = e^{(t-s)^\top R} \pi_s,$$

  or equivalently that $e^{-t^\top R} \pi_t$ is a martingale.
- On the event $\{t < \mu\}$, π_t equals some deterministic function $\psi(t)$.

Augmented Belief process : $(\pi_t, \xi_t)_{t\geq 0}$ belief process augmented with a dual process ξ, $\mathcal{F}^\pi$-measurable, and such that $\mathbb{E}[\xi_0] = y$ and

$$\mathbb{E}_{p,q}[\xi_t | \mathcal{F}_s^\pi] = e^{(t-s)(rI - Q)} \xi_s.$$

Verification result

Theorem

Define

$$\mathcal{H} := \{(p', y') \,|\, rV_*(p', y') - \vec{D}V_*(p', y'; {}^{\top}Rp', (rI - Q)y') \geq 0\}.$$

Let $y \in \partial_2^- V(p,q)$ *and* $\mu \in \mathcal{T}_r^X$ *such that there exists an augmented belief process* (π, ξ) *such that :*

(i) (π, ξ) *has bounded variation,*

(ii) $(\pi_t, \xi_t) \in \mathcal{H}$ *on* $\{t < \mu\}$,

(iii) *Jumps of* (π, ξ) *occur almost surely on "flat parts" of* V_*.

(iv) $V_*(\pi_\mu, \xi_\mu) = h_*(\pi_\mu, \xi_\mu)$ $\mathbb{P}'_p$*-a.s.,*

Then μ *is optimal, i.e.*

$$\inf_{\nu \in \mathcal{T}_r^Y} J_{p,q}(\mu, \nu) \geq V(p,q).$$

Remark : We do not ask for any additional regularity of V_*.

Plan

Let us solve a simple case.

Let us assume that :

- $K = L = \{0, 1\}$. $p, q \in [0, 1]$ denote the probability of 1.
- $R = Q = 0$ (constant Markov chains).
- $h = \begin{pmatrix} -4 & -2 \\ -1 & 1 \end{pmatrix}$, $f = \begin{pmatrix} -1 & 2 \\ 1 & 4 \end{pmatrix}$.

  $h(p, q) = 3p + 2q - 4$, $f(p, q) = 2p + 3q - 1$.

P1 would like to stop only in state $(X, Y) = (1, 1)$. He only observes X, and will never stop if $X = 0$ or if his "belief" on $Y = 1$ is below $1/2$.

P2 would like to stop only in state $(0, 0)$. He only observes Y, and will never stop if $Y = 1$ or if his "belief" on $X = 1$ is above $1/2$.

The Value function.

The value $V(p,q)$ of this game is the unique concave-convex Lipschitz function such that $h \leq V \leq f$ and
Subsolution : $\forall q \in [0,1]$, $\forall p \in Ext(V(.,q))$,

$$V(p,q) > h(p,q) \Rightarrow rV(p,q) \leq 0,$$

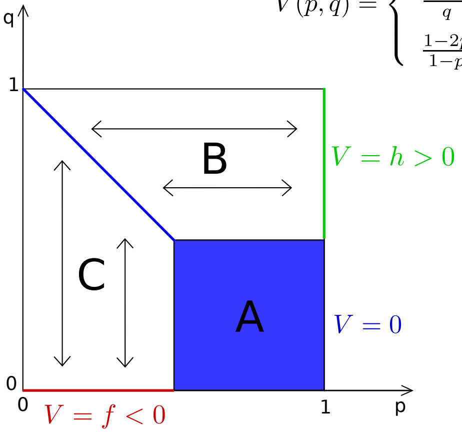
Supersolution : $\forall p \in [0,1]$, $\forall q \in Ext(V(p,.))$,

$$V(p,q) < f(p,q) \Rightarrow rV(p,q) \geq 0.$$

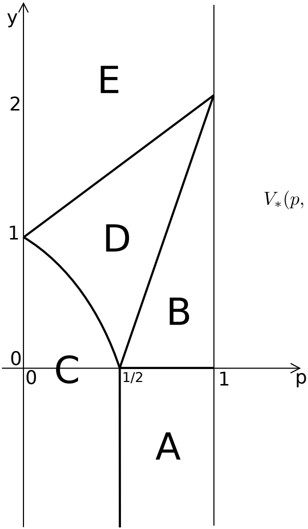
Using these properties, it is quite easy to compute explicitly V and then V_*.

The value function.

$$V(p,q) = \begin{cases} 0 & \text{if} \quad (A) \\ \frac{2q-1}{q}(p+q-1) & \text{if} \quad (B) \\ \frac{1-2p}{1-p}(p+q-1) & \text{if} \quad (C) \end{cases}$$



The conjugate function V_*.



$$V_*(p,y) = \begin{cases} 0 & \text{if} \quad (A) \\ \frac{1}{2}y & \text{if} \quad (B) \\ 1-2p & \text{if} \quad (C) \\ -2\sqrt{2-y}\sqrt{1-p}+3-2p & \text{if} \quad (D) \\ y-p & \text{if} \quad (E) \end{cases} .$$

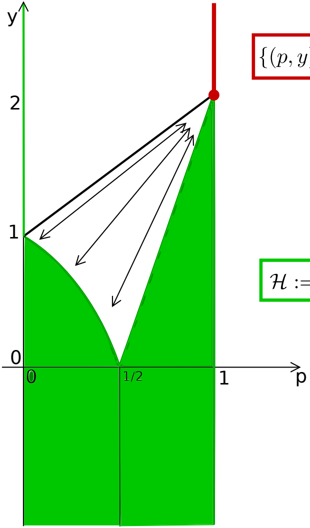
The Constraints for the augmented belief process.

In order to construct an optimal stopping time μ using our verification theorem, we have to find an augmented belief process (π, ξ) such that :

$$d\pi_t = d(\text{pure-jump-martingale}), \quad d\xi_t = r\xi_t dt + d(\text{pure-jump-martingale}).$$

- $(\pi_t, \xi_t) \in \mathcal{H}$ on $\{t < \mu\}$.
- $(\pi_\mu, \xi_\mu) \in \{V_* = h_*\}$.
- (π, ξ) jumps only on the "flat parts" of V_*.

The Constraints for the augmented belief process.



$\{(p,y) \,|\, V_*(p,y) = h_*(p,y)\}$

$\mathcal{H} := \{(p,y) \,|\, rV_*(p,y) - \vec{D}_2 V_*(p,y;ry) \geq 0\}$

Optimal Stopping time of player P1.

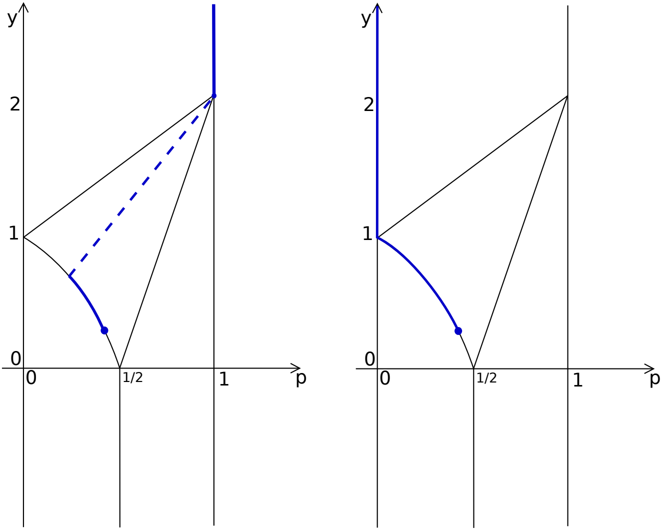
The optimal stopping time μ when starting from a point (p, q) with $0 < p < 1/2$ and $q = 1 - p$ is as follows :

- The starting dual variable is $y = \frac{\partial V}{\partial q}(p, q) = \frac{1-2p}{1-p}$.
- On some time-interval $[0, T_p]$, Player 1 stops with some intensity ρ_t conditionnaly on the fact that the state $X = 1$. As a result, the process π goes down continuously if $t < \mu$ and jumps to 1 whenever μ occurs. It reaches 0 at time T_p if μ did not occur (i.e. if $X = 0$). At time T_p, P1 has stopped with probability 1 if $X = 1$ (and 0 if $X = 0$).
- The density is given by :

$$\rho_t = r(\frac{1}{2} - p))e^{r(1-p)t}.$$

- The process ξ equals $\frac{1-2\pi_t}{1-\pi_t}$ if $t < \mu$ and jumps to 2 at μ.

Typical trajectories of the solution.



Plan

Work in progress.

- The analysis made for the example can be extended with more than two states and/or with non-zero transition matrices.
- **Conjecture :** One can always apply the verification argument.
- Extension (existence and variational characterization of the value) to a model with a public signal (diffusion with coefficients depending on (X, Y)).

Questions we would really like to answer some day...

- Extension to correlated chains (X, Y).
- Extension to diffusion processes (X, Y) (PDE in infinite-dimensional space)
- "Extension" to non-zero-sum games ?
- Can the construction of optimal stopping times be somehow adapted to differential games with integral payoffs ?

Thank you for your attention !