Ranking players in an ordinal coalitional framework

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Problem

- A company should rank some employees based on their job performance
- taking into account the ability of each employee to work alone on its own initiative and with others as a team
- Any attempt to evaluate all teams and single employees on a common quantitative scale turns out to be impossible

EXAMPLE (three employees 1, 2 and 3)

- The job performance of $\{1\}$ as a singleton coalition is significantly lower than the job performance of $\{3\}$
- the performance of the team $\{2,3\}$ is strictly lower than the performance of any other team (strong incompatibility between 2 and 3);
- $\{1,2\}$ is the most successful team. So:

$$\{1,2\} \succ \{3\} \succ \{1\} \succ \{2,3\}$$

Q: who is better between 1 and 3 (and should be promoted)?

Outline

Power relations and social rankings

- a total preorder $\succcurlyeq$ on 2^N as a coalitional power relation: for each $S, T \in 2^N$, $S \succcurlyeq T$ stands for 'S is considered at least as strong as T according to $\succcurlyeq$'.

- We call the map $\rho : \mathcal{P}^{2^N} \longrightarrow \mathcal{T}^N$, assigning to each power relation on 2^N a total binary relation on N, a *social ranking* solution. [Here $\mathcal{P}^{2^N}$ is the set of all power relations on 2^N, and $\mathcal{T}^N$ is the set of all total relations on N]

- Interpretation: for each $i, j \in N$, $i\rho(\succcurlyeq)j$ stands for 'i is considered at least as influential as j according to the social ranking ρ on $\succcurlyeq$'.

Is player i more influential than j?

We will say that i dominates j if for every possible coalition $S \subseteq N$ the number of coalitions at least as strong as S that contain i is larger than the number of those that contain j.

Axiom [DOM]

- We say that a social ranking satisfies DOM iff

$$i \text{ dominates } j \Rightarrow i\rho(\succcurlyeq)j$$

[and in addition $\neg(j\rho(\succcurlyeq)i)$ if the dominance holds strict for some coalition S]

Example

Consider the coalitional power relation

$$\{1,2,3\} \succ \{2\} \succ \{1,3\} \succ \{1,2\} \succ \{3\} \succ \{1\} \succ \emptyset \succ \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$
player 1	1	1	2	3	3	4	4	4
player 2	1	2	2	3	3	3	3	4
player 3	1	1	2	2	3	3	3	4

Note that both 1 and 2 dominate 3, whereas neither 1 dominates 2 nor 2 dominates 1.

Some notations on coalitional games

- A coalitional game (N, v): a finite *player set* $N = \{1, \ldots, n\}$ and a *characteristic function* $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$.

- the *Banzhaf value* (Banzhaf (1964)) of a coalitional game v on N:

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \in 2^{N \setminus \{i\}}} (v(S \cup \{i\}) - v(S)) \quad (1)$$

  for each $i \in N$

- Given a total preorder $\succcurlyeq$ on 2^N, we denote by $V(\succcurlyeq)$ the class of coalitional games that numerically represent $\succcurlyeq$ (i.e., $S \succcurlyeq T \Leftrightarrow v(S) \geq v(T)$ for each $S, T \in 2^N$ and each $v \in V(\succcurlyeq)$) and such that $v(\emptyset) = 0$.

Connection with the Banzhaf power index (1)

Theorem

Let $\succcurlyeq \in \mathcal{P}^{2^N}$. *For each* $i, j \in N$

$$i \text{ dominates } j \Leftrightarrow [\beta_i(v) \geq \beta_j(v) \text{ for every } v \in V(\succcurlyeq)].$$

$\Longrightarrow$ A social ranking that satisfies the DOM property is such that if i has more power than j according to the Banzhaf power index of every game representing $\succcurlyeq$, then i is ranked stronger than j.

Dichotomous total preorder

Consider the coalitional power relation

$$\{1,2,3\} \sim \{2\} \succ \{1,3\} \sim \{1,2\} \sim \{3\} \sim \{1\} \sim \emptyset \sim \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$
player 1	1	1	4	4	4	4	4	4
player 2	2	2	4	4	4	4	4	4
player 3	1	1	4	4	4	4	4	4

$\Longrightarrow$ In dichotomous power relations the dominance relation is total.

How to "decompose" a power relation

$$\{1,2,3\} \succ \{2\} \succ \{1,3\} \succ \{1,2\} \succ \{3\} \succ \{1\} \succ \emptyset \succ \{2,3\}$$

is the intersection of 7 dichotomous total preorders:

$$\{1,2,3\} \succ \{2\} \sim \{1,3\} \sim \{1,2\} \sim \{3\} \sim \{1\} \sim \emptyset \sim \{2,3\}$$

[here 1 dominates 2 and 2 dominates 1]

$$\{1,2,3\} \sim \{2\} \succ \{1,3\} \sim \{1,2\} \sim \{3\} \sim \{1\} \sim \emptyset \sim \{2,3\}$$

[here 2 dominates 1]

$$\{1,2,3\} \sim \{2\} \sim \{1,3\} \succ \{1,2\} \sim \{3\} \sim \{1\} \sim \emptyset \sim \{2,3\}$$

[here 1 dominates 2 and 2 dominates 1]

⋮

$$\{1,2,3\} \rhd \{2\} \sim \{1,3\} \rhd \{1,2\} \rhd \{3\} \rhd \{1\} \rhd \emptyset \rhd \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$	Sum
p. 1	1	2	2	3	3	4	4	4	23
p. 2	1	2	2	3	3	3	3	4	21

$$\{1,2,3\} \sqsupset \{2\} \sqsupset \{1,3\} \sqsupset \{1,2\} \sqsupset \{3\} \sqsupset \{1\} \sim \emptyset \sim \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$	Sum
p. 1	1	1	2	3	3	4	4	4	22
p. 2	1	2	2	3	3	4	4	4	23

The "relative scores" are $s_{12}(\trianglerighteq) = 23 - 21 = 2$ and $s_{21}(\sqsupseteq) = 23 - 22 = 1$.

"Additive" social ranking

- if a power relation $\succcurlyeq$ can be obtained as the intersection of two power relations $\trianglerighteq$ and $\sqsupseteq$, such that
- i is ranked better than j in the social ranking on $\trianglerighteq$
- and j is ranked better than i in the social ranking on $\sqsupseteq$,
- then the relative social ranking of i and j in $\succcurlyeq$ is determined by the comparison of the "relative scores" between i and j on $\trianglerighteq$ and on $\sqsupseteq$, respectively.

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- and j is ranked better than i in the social ranking on $\sqsupseteq$,
- then the relative social ranking of i and j in $\succcurlyeq$ is determined by the comparison of the "relative scores" between i and j on $\trianglerighteq$ and on $\sqsupseteq$, respectively.

Axiom [ADD] Let $\succcurlyeq \in \mathcal{P}^{2^N}$ and let $\trianglerighteq, \sqsupseteq \in K^{\succcurlyeq}$ be two power relations such that $i\rho(\trianglerighteq)j$ and $j\rho(\sqsupseteq)i$, for some $i, j \in N$.
A social ranking ρ satisfies the additivity property w.r.t. $\trianglerighteq$ and $\sqsupseteq$ if and only if the following relation holds:

$$s_{ij}(\trianglerighteq) \geq s_{ji}(\sqsupseteq) \Leftrightarrow i\rho(\trianglerighteq \cap \sqsupseteq)j.$$

$$\{1,2,3\} \rhd \{2\} \sim \{1,3\} \rhd \{1,2\} \rhd \{3\} \rhd \{1\} \rhd \emptyset \rhd \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$	Sum
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p. 1	1	1	2	3	3	4	4	4	22
p. 2	1	2	2	3	3	4	4	4	23

$s_{12}(\trianglerighteq) = 23 - 21 = 2$ and $s_{21}(\sqsupseteq) = 23 - 22 = 1 \Longrightarrow$ if a social ranking satisfies the DOM and the ADD (w.r.t. $\trianglerighteq$ and $\sqsupseteq$), then $1\rho(\succ)2$.

Connection with the Banzhaf power index (2)

Theorem

Let ρ be a social ranking which satisfies DOM and ADD w.r.t. $\trianglerighteq$ and $\sqsupseteq$. Then,

$$i\rho(\succcurlyeq)j \Leftrightarrow \beta_i(\hat{v}_{\trianglerighteq}) - \beta_j(\hat{v}_{\trianglerighteq}) \geq \beta_j(\hat{v}_{\sqsupseteq}) - \beta_i(\hat{v}_{\sqsupseteq}),$$

where $\hat{v}_{\trianglerighteq} \in V(\trianglerighteq)$ and $\hat{v}_{\sqsupseteq} \in V(\sqsupseteq)$ are the canonical games representing $\trianglerighteq$ and $\sqsupseteq$, respectively.

Here $\trianglerighteq$ and $\sqsupseteq$ are such that

$$\trianglerighteq \equiv \bigcap_{T \in \Theta^{ij}} \succcurlyeq_T \quad \text{and} \quad \sqsupseteq \equiv \bigcap_{T \in \Theta^{ji}} \succcurlyeq_T,$$

for some $i, j \in N$.
We call *canonical* game the coalitional game $\hat{v} \in V(\succcurlyeq)$ such that $\hat{v}(T) - \hat{v}(T^{\sigma}) = |[T]|$ (with $\hat{v}(\emptyset) = 0$), where $[T]$ is the indifference set of T wrt $\succcurlyeq$.

$$\{1,2,3\} \rhd \{2\} \sim \{1,3\} \rhd \{1,2\} \rhd \{3\} \rhd \{1\} \rhd \emptyset \rhd \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$
$\hat{v}_{\trianglerighteq}$	6	5	5	3	2	1	0	-1

$$\beta_1(\hat{v}_{\trianglerighteq})-\beta_2(\hat{v}_{\trianglerighteq}) = \frac{1}{2}(\hat{v}_{\trianglerighteq}(\{1\})-\hat{v}_{\trianglerighteq}(\{2\})+\hat{v}_{\trianglerighteq}(\{1,3\})-\hat{v}_{\trianglerighteq}(\{2,3\})) = 1$$

$$\{1,2,3\} \sqsupset \{2\} \sqsupset \{1,3\} \sqsupset \{1,2\} \sqsupset \{3\} \sqsupset \{1\} \sim \emptyset \sim \{2,3\}$$

	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$
$\hat{v}_{\sqsupseteq}$	5	4	3	2	1	0	0	0

$$\beta_2(\hat{v}_{\sqsupseteq})-\beta_1(\hat{v}_{\sqsupseteq}) = \frac{1}{2}(\hat{v}_{\sqsupseteq}(\{2\})-\hat{v}_{\sqsupseteq}(\{1\})+\hat{v}_{\sqsupseteq}(\{2,3\})-\hat{v}_{\sqsupseteq}(\{1,3\})) = \frac{1}{2},$$

Future directions (WP with M. Öztürk)

Given a power relation $\succcurlyeq$ on 2^N, maybe not all the comparisons are relevant.

$$\begin{array}{c} 1 \ vs \ 2 \\ \hline \{1\} \ vs \ \{2\} \\ \{1,3\} \ vs \ \{2,3\} \\ \{1,4\} \ vs \ \{2,4\} \\ \vdots \\ \{1,3,4\} \ vs \ \{2,3,4\} \\ \vdots \\ S \cup \{1\} \ vs \ S \cup \{2\} \\ \vdots \end{array}$$

for each $S \in 2^{N \setminus \{1,2\}}$.

Example

$N = \{1, 2, 3, 4\}$

$$1 \succ 2 \succ 3$$
$$13 \succ 23 \succ 12$$
$$24 \succ 14 \succ 34$$
$$1234 \sim 123 \sim 124 \sim 134 \sim 234$$

Relevant information:

1 vs. 2	2 vs. 3	1 vs. 3
$1 \succ 2$	$2 \succ 3$	$1 \succ 3$
$13 \succ 23$	$12 \prec 13$	$12 \prec 23$
$14 \prec 24$	$24 \succ 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$

Example

$N = \{1, 2, 3, 4\}$

$$1 \succ 2 \succ 3$$
$$13 \succ 12 \succ 23$$
$$24 \succ 14 \succ 34$$
$$1234 \sim 123 \sim 124 \sim 134 \sim 234$$

Relevant information:

1 vs. 2	2 vs. 3	1 vs. 3
$1 \succ 2$	$2 \succ 3$	$1 \succ 3$
$13 \succ 23$	$12 \prec 13$	$12 \succ 23$
$14 \prec 24$	$24 \succ 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$