

# Finite composite games: equilibria and dynamics

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This is a joint work with Cheng Wan, University of Oxford.

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We consider **finite games** :  
there are finitely many **“participants”**,  $i \in I$   
each of them has finitely many **“choices”**,  $p \in S^i$ .

The basic variable describing the interaction is thus a **profile**  
 $x = \{x^i, i \in I\}$ , where each  $x^i = \{x^{ip}, p \in S^i\}$  is an element of the  
**simplex**  $X^i = \Delta(S^i)$  on  $S^i$ . Let  $X = \prod_{i \in I} X^i$ .

We consider three frameworks with the following types of  
participants:

- (I) **populations of nonatomic players**,
- (II) **atomic splittable players**,
- (III) **atomic non splittable players**.

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We compare and unify the basic properties, expressed through variational inequalities, concerning equilibria, potential games and dissipative games, as well as associate evolutionary dynamics.

We extend the analysis to **composite games**.



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**Replicator dynamics** for one population

$S$  is the set of "types",  $x_t^p$  is the proportion of type  $p \in S$  in the population at time  $t$ ,  $A = ((A_{pq}))$  is the **fitness matrix**

$$\dot{x}_t^p = x_t^p [e^p A x_t - x_t A x_t], \quad p \in S$$

**Replicator dynamics** for two populations (cross matching)

$$\dot{x}_t^{1p} = x_t^{1p} [e^{1p} A^1 x_t^2 - x_t^1 A^1 x_t^2], \quad p \in S^1$$

and similarly for  $x^2$ .

**Replicator dynamics** for  $I$  populations

$$\dot{x}_t^{ip} = x_t^{ip} [A^i(e^{ip}, x_t^{-i}) - A^i(x_t^i, x_t^{-i})], \quad p \in S^i, i \in I$$

natural interpretation:  $x_t^{ip}, p \in S^i$ , is a mixed strategy of player  $i$ .

**Unilateral replicator dynamics** for one participant

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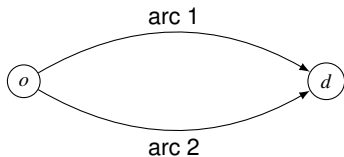
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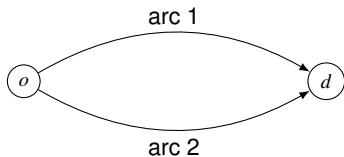
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## Routing game



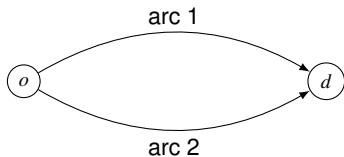
- Population games : each participant  $i \in I$  corresponds to a nonatomic set of agents having all the same characteristics.  $x^{ip}$  is the proportion of agents of choosing path  $p$  in population  $i$ . Two kinds of  $I$ -player games where each participant  $i \in I$  stands for an atomic player (with a given mass  $m^i$ ) :
- Splittable case:  $x^{ip}$  is the ratio that player  $i$  allocates to path  $p$ . (The set of pure moves of player  $i$  is  $X^i$ .)
- Non splittable case:  $x^{ip}$  is the probability that player  $i$  chooses path  $p$ . (The set of pure moves is  $S^i$  and  $x^i$  is a mixed strategy.)

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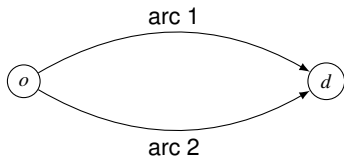
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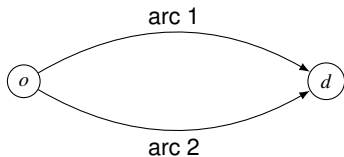


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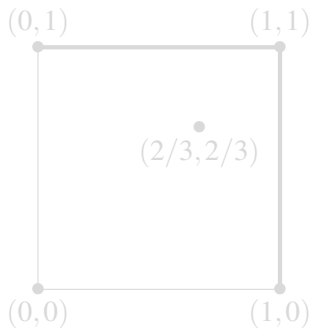
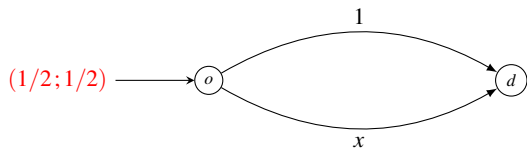


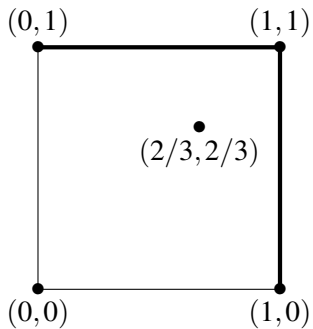
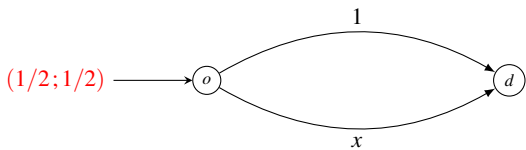
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## Framework I: population games

The payoffs are defined by a family of continuous functions  $\{F^{ip}, i \in I, p \in S^i\}$ , all from  $X$  to  $\mathbb{R}$ , where  $F^{ip}(x)$  is the outcome of a member in population  $i$  choosing  $p$ , when the environment is given by  $x$ .

An **equilibrium** is a point  $x \in X$  satisfying:

$$x^{ip} > 0 \Rightarrow F^{ip}(x) \geq F^{iq}(x), \quad \forall p, q \in S^i, \forall i \in I. \quad (1)$$

This corresponds to a **Wardrop equilibrium**.

An equivalent characterization of (1) is through the variational inequality:

$$\langle F^i(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i, \forall i \in I, \quad (2)$$

or alternatively:

$$\langle F(x), x - y \rangle = \sum_{i \in I} \langle F^i(x), x^i - y^i \rangle \geq 0, \quad \forall y \in X. \quad (3)$$

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## Framework II: atomic splittable

In this case each participant  $i \in I$  corresponds to an atomic player with action set  $X^i$ . Given functions  $F^{ip}$  as introduced above, his gain is defined by:

$$H^i(x) = \langle x^i, F^i(x) \rangle = \sum_{p \in S^i} x^{ip} F^{ip}(x).$$

An **equilibrium** is as usual a profile  $x \in X$  satisfying:

$$H^i(x) \geq H^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \forall i \in I. \quad (4)$$

Suppose that for all  $p \in S^i$ ,  $F^{ip}(\cdot)$  is of class  $\mathcal{C}^1$  on a neighborhood  $\Omega$  of  $X$ , then any solution of (4) satisfies

$$\langle \nabla H(x), x - y \rangle = \sum_{i \in I} \langle \nabla^i H^i(x), x^i - y^i \rangle \geq 0, \quad \forall y \in X. \quad (5)$$

where  $\nabla^i$  is the gradient w.r.t.  $x^i$ . Moreover, if each  $H^i$  is concave with respect to  $x^i$ , there is equivalence.



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## Framework III: atomic non splittable

We consider here an  $I$ -player game where the payoff is defined by a family of functions  $\{G^i, i \in I\}$  from (the finite set)  $S = \prod_{i \in I} S^i$  to  $\mathbb{R}$ .

We still denote by  $G$  the multilinear extension to  $X$  where each  $X^i = \Delta(S^i)$  is considered as the set of mixed actions of player  $i$ .

An **equilibrium** is a profile  $x \in X$  satisfying:

$$G^i(x^i, x^{-i}) \geq G^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \forall i \in I. \quad (6)$$

Let  $VG^i$  denote the vector payoff associated to  $G^i$ . Explicitly,  $VG^{ip} : X^{-i} \rightarrow \mathbb{R}$  is defined by  $VG^{ip}(x^{-i}) = G^i(p, x^{-i})$ , for all  $p \in S^i$ . Hence  $G^i(x) = \langle x^i, VG^i(x^{-i}) \rangle$ .

An equilibrium is thus a solution of :

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# Variational inequality

Note that  $F$ ,  $\nabla H$  and  $VG$  play similar roles in the three frameworks.

We call them **evaluation functions** and denote them by  $\Phi$ .

## Definition

$NE(\Phi)$  is the set of  $x \in X$  satisfying:

$$\langle \Phi(x), x - y \rangle \geq 0, \quad \forall y \in X. \quad (8)$$

$NE(\Phi)$  = equilibria of  $\Gamma(\Phi)$ .

Let  $C \subset \mathbb{R}^d$  be a closed convex set and  $\Psi$  a map from  $C$  to  $\mathbb{R}^d$ . Consider the variational inequality:

$$\langle \Psi(x), x - y \rangle \geq 0, \quad \forall y \in C. \quad (9)$$

Four equivalent representations are given by:

$$\Psi(x) \in N_C(x), \quad (10)$$

where  $N_C(x)$  is the normal cone to  $C$  at  $x$ ;

$$\Psi(x) \in [T_C(x)]^\perp, \quad (11)$$

where  $T_C(x)$  is the tangent cone to  $C$  at  $x$  and  $[T_C(x)]^\perp$  its polar;

$$\Pi_{T_C(x)} \Psi(x) = 0, \quad (12)$$

where  $\Pi$  is the projection operator on a closed convex subset; and

$$\Pi_C[x + \Psi(x)] = x. \quad (13)$$

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# Potential games

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$$\langle \nabla^i W(x) - \mu^i(x) \Phi^i(x), y^i \rangle = 0, \quad \forall x \in X, \forall y^i \in X_0^i, \forall i \in I, \quad (14)$$

where  $X_0^i = \{y \in \mathbb{R}^{|S^i|}, \sum_{p \in S^i} y_p = 0\}$  is the tangent space to  $X^i$ . The game  $\Gamma(\Phi)$  is then called a **potential game** and one says that  $\Phi$  **derives** from  $W$ .

Monderer and Shapley, Sandholm

## Theorem

*Let  $\Gamma(\Phi)$  be a game with potential  $W$ .*

- 1. Every local maximum of  $W$  is an equilibrium of  $\Gamma(\Phi)$ .*
- 2. If  $W$  is concave on  $X$ , then any equilibrium of  $\Gamma(\Phi)$  is a global maximum of  $W$  on  $X$ .*



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The game  $\Gamma(\Phi)$  is *dissipative* if  $\Phi$  satisfies:

$$\langle \Phi(x) - \Phi(y), x - y \rangle \leq 0, \quad \forall (x, y) \in X \times X.$$

In the framework of population games, Hofbauer and Sandholm studied this class under the name “stable games”.

Let  $SNE(\Phi)$  be the set of  $x \in X$  satisfying:

$$\langle \Phi(y), x - y \rangle \geq 0, \quad \forall y \in X.$$

## Proposition

If  $\Gamma(\Phi)$  is *dissipative*

$$SNE(\Phi) = NE(\Phi).$$

in particular  $NE(\Phi)$  is convex.

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$$\langle \Phi(y), x - y \rangle \geq 0, \quad \forall y \in X.$$

## Proposition

If  $\Gamma(\Phi)$  is *dissipative*

$$SNE(\Phi) = NE(\Phi).$$

*in particular*  $NE(\Phi)$  is convex.

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**Dynamics**

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## Definitions

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where for each  $i \in I$ ,  $\mathcal{B}_\Phi^i(x) \in X_0^i$  and  $X$  is invariant.

Replicator dynamics (RD) (Taylor and Jonker)

$$\dot{x}_t^{ip} = x_t^{ip} [\Phi_t^{ip}(x_t) - \bar{\Phi}^i(x_t)], \quad p \in S^i, i \in I,$$

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$$\dot{x}_t^i = \Pi_{T_{X^i}(x_t^i)}[\Phi^i(x_t)], \quad i \in I,$$

where we recall that  $T_{X^i}(x^i)$  denotes the tangent cone to  $X^i$  at  $x^i$ .

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$$\dot{x}_t^i = \Pi_{X^i}[x_t^i + \Phi^i(x_t)] - x_t^i, \quad i \in I.$$

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# General properties

We define here properties expressed in terms of  $\Phi$ .

## Definition

Dynamics  $\mathcal{B}_\Phi$  satisfies:

i) **positive correlation (PC)**(Sandholm) if:

$$\langle \mathcal{B}_\Phi^i(x), \Phi^i(x) \rangle > 0, \quad \forall i \in I, \forall x \in X \text{ s.t. } \mathcal{B}_\Phi^i(x) \neq 0.$$

This corresponds to MAD (myopic adjustment dynamics, Swinkels)

ii) **Nash stationarity** if:

for  $x \in X$ ,  $\mathcal{B}_\Phi(x) = 0$  if and only if  $x \in NE(\Phi)$ .

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## Proposition

*All previous dynamics (RD), (BNN), (Smith), (LP), (GP) and (BR) satisfy (PC).*

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*(BNN), (Smith), (LP), (GP) and (BR) satisfy Nash stationarity on  $X$ .*

*(RD) satisfy Nash stationarity on  $\text{int} X$ .*

## Proposition

*Consider a potential game  $\Gamma(\Phi)$  with potential function  $W$ . If the dynamics  $\dot{x} = \mathcal{B}_\Phi(x)$  satisfies (PC), then  $W$  is a strict Lyapunov function for  $\mathcal{B}_\Phi$ . Besides, all  $\omega$ -limit points are rest points of  $\mathcal{B}_\Phi$ .*

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It follows that, with the appropriate definitions, the convergence results established for several dynamics and potential games in framework **I** extend. Explicitly:

### Proposition

*Consider a potential game  $\Gamma(\Phi)$  with potential function  $W$ .*

*If the dynamics is (RD), (BNN), (Smith), (LP), (GP) or (BR),  $W$  is a strict Lyapunov function for  $\mathcal{B}_\Phi$ .*

*In addition, except for (RD), all  $\omega$ -limit points are equilibria of  $\Gamma(\Phi)$ .*

Similar results hold for dissipative games with ad hoc Lyapunov functions.

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# Congestion games and composite games

In a network congestion game, or routing game, the underlying network is a finite directed graph  $G = (V, A)$ , where  $V$  is the set of nodes,  $A$  the set of links.

$\mathbf{l} = (l_a)_{a \in A}$  denotes a family of cost functions from  $\mathbb{R}$  to  $\mathbb{R}^+$ : if the aggregate weight on arc  $a$  is  $m$ , the cost per unit (of weight) is  $l_a(m)$ .

The set  $I$  of participants is finite. A participant  $i$  is characterized by his *weight*  $m^i$  and an *origin/destination pair*  $(o^i, d^i) \in V \times V$  such that the constraint is to send a quantity  $m^i$  from  $o^i$  to  $d^i$ .

The set of choices of participant  $i \in I$  is  $S^i$ : a family of directed acyclic paths linking  $o^i$  to  $d^i$ . Let  $P = \cup_{i \in I} S^i$ .

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In each of the three frameworks considered, a configuration  $x$  induces a (random) **flow**  $f$  on the arcs. This defines the cost on each arc then for each path and finally the payoff of each participant.

Congestion games are thus natural settings where each kind of participants appears.

Moreover one can even consider a game where participants of different natures coexist: some of them being of type **I**, **II** or **III**. This leads to the notion of **composite game**.

Composite congestion games with participants of type **I** and **II** have been studied by Harker; Boulogne Altman, Pourtallier and Kameda; Yang and Zhang; Cominetti, Correa and Stier-Moses, etc... under the name "mixed equilibria".

In addition, congestion games are a natural example of an aggregative game (Selten) where the payoff of a participant  $i$  depends only on  $x^i \in X^i$  and on some fixed dimensional function  $\alpha^i(\{x^j\}_{j \neq i}) \in \Delta(\mathbb{R}^P)$ .

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# Composite games

We have seen that the properties of equilibrium and dynamics in the three frameworks all depend on the evaluation function  $\Phi$  and the variational inequalities associated to it. One can define a more general class of games called **composite games**, which exhibit different types of players.

Explicitly consider a finite set  $I_1$  of populations composed of nonatomic players, a finite set  $I_2$  of atomic splittable players and a finite set  $I_3$  of atomic non splittable players. Let  $I = I_1 \cup I_2 \cup I_3$ .

All the analysis of the previous sections extend to these configurations where  $x = \{x^i\}_{i \in I_1 \cup I_2 \cup I_3}$  and  $\Phi^{ip}(x)$  depends upon the type of participant  $i$ :

- expression of equilibria through variational inequalities,
- definition of potential games and dissipative games,
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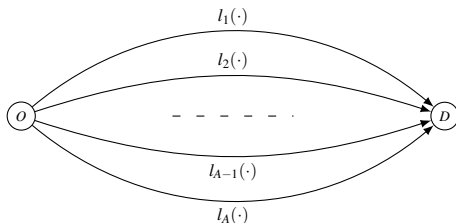
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- definition of potential games and dissipative games,
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## One example of a composite potential game

Consider a composite congestion game, with three types of participants  $i \in I = I_1 \cup I_2 \cup I_3$ , of mass  $m^i$  each, taking place in a network composed of two nodes  $o$  and  $d$  connected by a finite set  $A$  of parallel arcs.

Figure: Example of a composite potential game



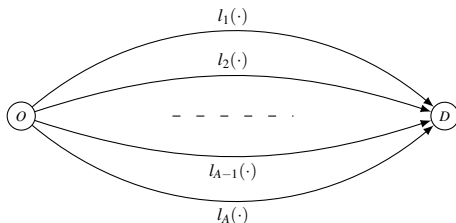
Denote by  $s = (s^k)_{k \in I_3} \in S_3 = A^{I_3}$  a pure strategy profile of participants in  $I_3$  and let  $z = ((x^i)_{i \in I_1}, (x^j)_{j \in I_2}, (s^k)_{k \in I_3})$ . Let  $f(z)$  be the aggregate flow induced by the pure-strategy profile  $z$ .

Namely:  $f_a(z) = \sum_{i \in I_1} m^i x_a^i + \sum_{j \in I_2} m^j x_a^j + \sum_{k \in I_3} m^k \mathbf{1}_{\{s^k = a\}}$

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## Theorem

Assume that for all  $a \in A$ , the per-unit cost function is affine, i.e.  $l_a(u) = b_a u + d_a$ , with  $b_a > 0$  and  $d_a \geq 0$ . Then a composite congestion game on this network is a potential game.

A potential function defined on  $X$  is given by:

$$W(x) = - \sum_{s \in S_3} \left( \prod_{k \in I_3} x_{s^k}^k \right) \left\{ \frac{1}{2} \sum_{a \in A} b_a [(f_a(z))^2 + \sum_{j \in I_2} (m^j x_a^j)^2 + \sum_{k \in I_3} (m^k)^2 \mathbf{I}_{\{s^k=a\}}] + \sum_{a \in A} d_a f_a(z) \right\},$$

with  $\mu^i(x) \equiv m^i$  for all  $i \in I = I_1 \cup I_2 \cup I_3$  and all  $x \in X$ .

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## Related topics

**Asymptotic analysis for aggregative games** (Haurie and Marcotte)

Replace one participant (atomic)  $i$  of size  $m_i$  by  $n$  participants with same characteristics and weight  $m_i/n$ . Accumulation points of a sequence of equilibria as  $n$  goes to  $\infty$  are equilibria in the game where participant  $i$  is a population.

Composite players

A composite player of weight  $m^i$  is described by the splittable component of weight  $m^{i,0}$  and the non splittable components of weight  $m^{i,l}$  hence represented by a vector

$\underline{m}^i = (m^{i,0}, m^{i,1}, \dots, m^{i,n^i})$ , where  $n^i \in \mathbf{N}^*$ ,  $m^{i,0} \geq 0$ ,  $m^{i,l} > 0$  and  $m^{i,0} + \sum_{l=1}^{n^i} m^{i,l} = m^i$ . Player  $i$  may allocate proportions of the splittable component to different choices and also allocate different non splittable components to different choices. However, a non splittable component cannot be divided.

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## Delegation games

In the splittable case (or more generally for a composite player) a player  $i$  can delegate his mass among several players and get as payoff the sum of the payoff of the delegates.

- conditions to have simple best reply strategies
- dynamical stability

## Reinforcement and learning

Starting from a discrete time random adjustment process, tools from stochastic approximation may allow to work with a continuous time deterministic dynamics

However the state variable may change: in fictitious play

$x_{n+1} \in BR(\bar{x}_n)$  leads to  $\dot{z}_t \in BR(z_t) - z_t$  but now  $z^i \in X^i$

corresponds to the time average behavior of participant  $i$ .



## Delegation games

In the splittable case (or more generally for a composite player) a player  $i$  can delegate his mass among several players and get as payoff the sum of the payoff of the delegates.

- conditions to have simple best reply strategies
- dynamical stability

## Reinforcement and learning

Starting from a discrete time random adjustment process, tools from stochastic approximation may allow to work with a continuous time deterministic dynamics

However the state variable may change: in fictitious play

$x_{n+1} \in BR(\bar{x}_n)$  leads to  $\dot{z}_t \in BR(z_t) - z_t$  but now  $z^i \in X^i$

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SORIN S., WAN C. *Finite composite games: equilibria and dynamics*, ArXiv:1503.07935v1, 2015.