# Cheap talk and commitment in Bayesian games

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## Motivation

- Study cooperation in noncooperative games with asymmetric information: the noncooperative game describes the basic interactive decision problem, the "default game."

- Account for the following scenario: players exchange information by just talking together and then agree on a binding agreement.

- Make (a reverse) use of the "Folk theorem": from noncooperative solutions of the infinitely repeated game to cooperative solutions of the one-shot game.

## Related literature

- Cooperation/binding agreements in noncooperative games: Nash (1953), Myerson (1984, 1991, ....), Peters and Szentes (2012), Celik and Peters (2011, 2015), A. Kalai and E. Kalai (2013), etc.

- "Folk theorem" in one-shot games: A. Kalai, E. Kalai, Lehrer and Samet (2010), Myerson (1991), Tennenholtz (2004), Forges (2013), etc.

- Insights from infinitely repeated games to study communication and/or cooperation in one-shot games: Aumann, Maschler and Stearns (1968), Forges (1990), Aumann and Hart (2003), Simon, Spież and Toruńczyk (2008), etc.

**MODEL: Two-person one-shot Bayesian game $B(p)$**

$K$: finite set of types of player 1 (the only informed player)

$p \in \Delta(K)$: probability distribution over $K$, $p^k > 0$ for every $k \in K$.

Player 1's type $k$ is chosen in $K$ according to $p$ at a virtual initial stage of the game; only player 1 is informed of $k$.

$A_i$: finite set of actions of player $i$, $i = 1, 2$.

Player 1 and player 2 simultaneously choose an action in $A_1$ and $A_2$ respectively.

Their respective payoffs are $U^k(a_1, a_2)$ and $V^k(a_2)$ when player 1's type is $k$ and $a \in A = A_1 \times A_2$ is chosen.

Player 2's payoff does not depend on player 1's action (as in Sender-Receiver games).

**DEFINITION of a COOPERATIVE SOLUTION**

- as a Nash equilibrium of a noncooperative game (implementation)

- by a list of desirable properties (characterization)

then sufficient conditions for **EXISTENCE**

Given $p \in \Delta(K)$, a set of signals $S$ and a decision mapping $\chi : S \to \Delta(A)$, let $G(p, S, \chi)$ be the following game:

- Player 1's type $k$ is chosen in $K$ according to $p$, player 1 is informed of $k$.
- Player 1 sends a signal $s \in S$ to player 2.
- The joint decision $\chi(s) \in \Delta(A)$ is proposed to both players.
- The players simultaneously accept or reject $\chi(s)$.
- If they both accept $\chi(s)$, player 1 gets $U^k(\chi(s))$ and player 2 gets $V^k(\chi(s))$. If at least one of them rejects $\chi(s)$, player 1 chooses $a_1$, player 2 chooses $a_2$, player 1 gets $U^k(a_1, a_2)$ and player 2 gets $V^k(a_2)$.

**A solution of** $B(p)$ by "cheap talk and commitment submitted to unanimous approval" consists of $(S, \mu, \chi)$ :

- $S$: finite set of signals

- $\mu : K \to \Delta(S)$ : signalling strategy for player 1; $\mu(s \mid k) :=$ probability of sending $s$ given type $k$.

- $\chi : S \to \Delta(A)$ : joint decision mapping; $\chi(s) :=$ lottery over $A$ that is performed on behalf of the players.

such that $\mu$ and unanimous approval of $\chi(s)$ for every $s \in S$ is part of a perfect Bayesian equilibrium (PBE) of the game $G(p, S, \chi)$.

## Characterization of solutions

$(S, \mu, \chi)$ is a solution of $B(p)$ by "cheap talk and commitment submitted to unanimous approval" $\Leftrightarrow$

**(i)** $\mu$ is *incentive compatible* given $\chi$.

**(ii)** $\chi$ is *optimal* for player 2 given $\mu$.

**(iii)** Player 1's interim expected payoff from $(S, \mu, \chi)$ is *individually rational*.

These conditions look familiar, but their meaning here is not the same as in other frameworks. In particular, player 2's commitment is limited so that *no revelation principle* holds; incentive compatibility is thus very demanding.

## (i) Incentive compatibility (I.C.)

Player 1 sends his signal by himself, without the help of a mediator; hence he randomizes over signals $s, s' \in S$ if and only he is indifferent between $s$ and $s'$.

$\mu$ is *incentive compatible* (given $\chi$) iff

$\forall k \in K$, $\forall s, s' \in S$ : $\mu(s \mid k) > 0$ and $\mu(s' \mid k) > 0$

$\Rightarrow U^k(\chi(s)) = U^k(\chi(s'))$

$\forall k \in K$, $\forall s, s' \in S$ : $\mu(s \mid k) > 0$ and $\mu(s' \mid k) = 0$

$\Rightarrow U^k(\chi(s)) \geq U^k(\chi(s'))$

Similar conditions appear in Aumann, Maschler and Stearns (1968), Sorin (1983), Hart (1985), Aumann and Hart (2003).

## (ii) Optimality for player 2

A mixed decision $\tau \in \Delta(A_2)$ is optimal for player 2 given his belief $q \in \Delta(K)$ if $\sum_{k \in K} q^k V^k(\tau) = \max_{a_2 \in A_2} \sum_{k \in K} q^k V^k(a_2)$.

Let $R(q) = \{\tau \in \Delta(A_2) : \tau \text{ is optimal given } q\}$ and let $R = \cup_{q \in \Delta(K)} R(q)$.

$R$ is the set of player 2's strategies that can be *rationalized*, in the sense that they are optimal for some belief.

The prior probability $p$ over $K$ and a signalling strategy $\mu : K \to \Delta(S)$ induce posterior probability distributions $p_s(\mu)$ over $K$ for every signal $s \in S$ (assuming wlog that $\forall s \in S$, $\exists k : \mu(s \mid k) > 0$).

$\chi$ is *optimal* for player 2 given $\mu$ iff

for every $s \in S$, $\chi(s)$ is optimal given $p_s(\mu) \Leftrightarrow$ *marg* $_{A_2}(\chi(s)) \in R(p_s(\mu))$

## (iii) Individual rationality (I.R.) for player 1

Player 1's interim expected payoff $(u^k)_{k\in K}$ is *individually rational* iff player 2 has a rationalizable strategy that prevents every type $k$ of player 1 from getting more than $u^k$, namely

$$\exists \tau \in R \; \forall k \in K \; \forall a_1 \in A_1 : U^k(a_1, \tau) \leq u^k.$$

If $\mu$ is $I.C.$ given $\chi$, then, for every type $k$, all signals $s$ such that $\mu(s \mid k) > 0$ lead to the *same* payoff $U^k(\chi(s)) \equiv u^k$.

In a solution $(S, \mu, \chi)$, I.R. for player 1 can thus be interpreted as a *posterior* I.R. condition.

**Does every Bayesian game $B(p)$ have a solution?**

**Example:**

$$(U^1, V^1) = \begin{array}{ccc} & \ell & r \\ t & 1,1 & 2,0 \\ b & 0,1 & 0,0 \end{array} \qquad (U^2, V^2) = \begin{array}{ccc} & \ell & r \\ t & 0,0 & 0,1 \\ b & 2,0 & 1,1 \end{array}$$

$R = \Delta(\{\ell, r\})$

$(u^1, u^2)$ is I.R. for player 1 $\Leftrightarrow u^1 + u^2 \geq 3$.

For every $x \in \Delta(A)$, $U^1(x)+ U^2(x) \leq 2$.

Let $s$ such that $\mu(s \mid 1) > 0$ and $\mu(s \mid 2) > 0$; I.C. $\Rightarrow u^k = U^k(\chi(s))$, $k = 1, 2$, with $\chi(s) \in \Delta(A) \Rightarrow u$ cannot be I.R. for player 1.

$$
(U^1, V^1) = \begin{array}{ccc} & \ell & r \\ t & 1,1 & 2,0 \\ b & 0,1 & 0,0 \end{array} \qquad (U^2, V^2) = \begin{array}{ccc} & \ell & r \\ t & 0,0 & 0,1 \\ b & 2,0 & 1,1 \end{array}
$$

Only way to possibly get I.R. for player 1: *completely revealing* solution. Then optimality for player 2 $\Rightarrow \chi(\ell \mid 1) = 1$, $\chi(r \mid 2) = 1 \Rightarrow$ player 1 has at best $(1, 1)$, which is not I.R.

$\Rightarrow$ **no solution**, even if PBE of $G(p, S, \chi)$ is weakened to NE.

I.R. for player 1 ($u^1 + u^2 \geq 3$) is much more demanding than "I.R. for every type" ($u^1 \geq 1$ and $u^2 \geq 1$).

The allocation $\delta(1) = (t, \ell)$, $\delta(2) = (b, r)$ is I.C., optimal for player 2 and I.R. for every type of player 1, but cannot be implemented by cheap talk and commitment submitted to unanimous approval.

**Assumptions guaranteeing the existence of a cooperative solution**

*I. No decision for the informed player, Sender-Receiver game:*

Nonrevealing solution.

*II. Rationalizable uniform punishment strategy (RUPS):*

Let for every $k \in K$, $\underline{m}^k = \min_{\tau \in R} \max_{a_1 \in A_1} U^k(a_1, \tau)$.

$\tau^* \in R$ *is a rationalizable uniform punishment strategy (RUPS) of player 2 iff* $\forall k \in K \; \forall a_1 \in A_1 : U^k(a_1, \tau^*) \leq \underline{m}^k$.

Under RUPS, $u = (u^k)_{k \in K}$ is I.R. for player 1 iff $u^k \geq \underline{m}^k \; \forall k \in K$.

The example shows that RUPS is crucial for the "if" part.

**Possible proof of existence of a solution under RUPS**

Apply results that were initially conceived to establish the existence of Nash equilibria in undiscounted two-person infinitely repeated games with a single informed player:

Sorin (1983) in the case of two types (proof using elementary mathematical tools), Simon, Spież and Toruńczyk (1995), Renault (2000), Simon (2002).

Simon, Spież and Toruńczyk (2008) explicitly proposes an application to one-shot decision problems.

**Theorem (**Renault (2000), Simon (2002), Simon et al. (2008))

*Fix a finite set $K$, a compact, convex set $X$, linear functions $U^k : X \longrightarrow \mathbb{R}$, $k \in K$, a lower-semi-continuous function $f : \Delta(K) \longrightarrow \mathbb{R}$ and a non-empty convex valued, upper-hemi-continuous correspondence $F : \Delta(K) \longrightarrow X$ s.t.*

Assumption: $\forall q, \rho \in \Delta(K)\ \exists\, x$ *such that* $x \in F(q)$ *and* $\sum_k \rho^k U^k(x) \geq f(\rho)$

*Then there exist $S$ and for every $p$, $\mu : K \longrightarrow \Delta(S)$ and $\chi : S \longrightarrow X$ s.t.*

**(i)** $\mu$ is *incentive compatible* given $\chi$; *so,* $u^k = U^k(\chi(s))$ *if* $\mu(s \mid k) > 0$.

**(ii)** $\forall s \in S : \chi(s) \in F(p_s)$.

**(iii)** $\forall q \in \Delta(K) : q \cdot u \geq f(q)$.

**(ii):** Let $F(q) = \left\{x \in \Delta(A) : x \text{ is optimal given } q, \text{ i.e., } \textit{marg}_{A_2}(x) \in R(q)\right\}$.
Then $\chi(s) \in F(p_s)$ for every $s \in S \Leftrightarrow \chi$ is *optimal* for player 2 given $\mu$.

**(iii):** Let for every $q \in \Delta(K)$, $f(q) = \min_{\tau \in R} \max_{a_1 \in A_1} \sum\limits_k q^k U^k(a_1, \tau)$.
Under RUPS, $u = (u^k)_{k \in K}$ is I.R. for player 1 $\Leftrightarrow \forall q \in \Delta(K) : q \cdot u \geq f(q)$.

**Assumption:** Let $q, \rho \in \Delta(K)$.

Let $\tau(q) \in R(q)$ and $\widehat{a}_1 \equiv \widehat{a}_1(\rho, q) \in \arg\max \sum_k \rho^k U^k(\cdot, \tau(q))$.

Take $\widehat{x} \equiv \widehat{x}(\rho, q) = \widehat{a}_1(\rho, q) \otimes \tau(q)$; $\widehat{x} \in F(q)$;

$\sum_k \rho^k U^k(\widehat{x}) = \sum_k \rho^k U^k(\widehat{a}_1, \tau(q)) \geq f(\rho)$.

**Is there a simpler proof?** There may be no "simple" solution...

**Example:** $U^1(\cdot) = \begin{array}{cccc} & \ell & c & r \\ t & 1 & 1 & 1 \\ b & 0 & 0 & 3 \end{array} \qquad U^2(\cdot) = \begin{array}{cccc} & \ell & c & r \\ t & 3 & 0 & 0 \\ b & 2 & 2 & 2 \end{array}$

$\underline{m}^1 = 1$, $\underline{m}^2 = 2$, $c$ is a RUPS of player 2.

Let $q \in [0,1]$ be the probability that $k = 1$. Choose player 2's payoffs so that

$$R(q) = \begin{array}{lll} r & if & 0 \leq q \leq \frac{1}{4} \\ c & if & \frac{1}{4} \leq q \leq \frac{3}{4} \\ \ell & if & \frac{3}{4} \leq q \leq 1 \end{array}$$

At $p = \frac{1}{2}$: there is **NO** *nonrevealing* solution, **NO** *completely revealing* solution but there is a *partially revealing* solution.

Player 1 sends signals $y$ and $g$ so as to reach the posteriors $p_y = \frac{1}{4}$ and $p_g = \frac{3}{4}$.

Let then $x_y = b \otimes (\frac{2}{3}c + \frac{1}{3}r)$ and $x_g = t \otimes (\frac{2}{3}\ell + \frac{1}{3}c)$.

For player 1, $U^1(x_y) = U^1(x_g)$, $U^2(x_g) = U^2(x_y)$.

For player 2, any mixture of $c$ and $r$ (resp., of $\ell$ and $c$) is optimal at $p_y = \frac{1}{4}$ (resp., $p_g = \frac{3}{4}$).

Here, the payoffs at the partially revealing solution are exactly individually rational. There are more interesting examples, in which payoffs at the solution improve upon Bayesian Nash equilibrium payoffs.

**Conclusion and extensions**

- Under strong conditions (single informed player, uninformed player's payoff independent of informed player's action, RUPS), there exists a solution by "cheap talk and commitment submitted to unanimous approval"; existence may fail if RUPS does not hold.

- Extension to the case of general payoffs ??? Yes, if the definition of a solution is weakened by requiring implementation in NE instead of PBE; then RUPS can be weakened to UPS.

- Even with implementation in NE, the result does not extend to the case of lack of information on both sides, with private values and UPS.