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Epsilon-Nash equilibrium of a Multi Player NZSDG in discrete time

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Outline of the talk

I- Motivations and presentation of the problem
- Preliminary notations and hypotheses
- The multi-player non zero sum Dynkin game (NZSDG)

II- Solving the problem

III- Conclusion

Introduction : Setting and notations

- Discrete setting : $(\Omega, \mathbb{F}, \mathbb{P})$, on which is defined $\mathcal{F} := (\mathcal{F}_t)_{t \in \mathbb{N}}$
- There exists N players $\pi_1,\ \pi_2, \cdots \pi_N$ s.t.
  • If each π_i stops at time τ_i, the game terminates at random time $R(\omega) := \tau_1 \wedge \cdots \wedge \tau_n$,
  • $I = \{j \in \{1, \cdots, N\}, \tau_j = R(\omega)\}$:= (random) coalition of players stopping the game.
- Objective : For each i, π_i tries to optimize its payoff

$$J_i(\tau_1, \cdots \tau_N) = X^{i, I(\omega)}_{R(\omega)},$$

(among all stopping times $\tau_1, \cdots \tau_N$:= control actions)

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Introduction : questions to solve

- Does there exists an Nash equilibrium point (NEP) for the game (resp. an ϵ NEP) ?
  A NEP for the Dynkin game is a N tuple $(\tau_1^*, \cdots, \tau_N^*)$ of stopping times satisfying that, for each i and whatever $\tau_i \neq \tau_i^*$,

$$\begin{aligned} &\mathbb{E}\left(J_i(\tau_1^*, \cdots \tau_{i-1}^*, \tau_i, \tau_{i+1}^*, \cdots \tau_N^*)\right) \\ &\qquad\qquad \leq \mathbb{E}\left(J_i(\tau_1^*, \cdots, \tau_{i-1}^*, \tau_i^*, \tau_{i+1}^*, \cdots, \tau_N^*)\right) \\ &(\text{resp.} \qquad\qquad \leq \mathbb{E}\left(J_i(\tau_1^*, \cdots, \tau_{i-1}^*, \tau_i^*, \tau_{i+1}^*, \cdots, \tau_N^*) + \epsilon\right)). \end{aligned}$$

- Is the NEP (or the ϵ NEP) unique ?
- How many players choose to stop the game at optimality ?

Presentation of the problem

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Introduction : list of hypotheses

- No regularity of the payoff processes (such as supermartingale property)
- Let $\bar{\mathbb{N}} = \mathbb{N} \bigcup \{\infty\}$, each payoff process $(X_t^{i,I})_{t\in\mathbb{N}}$ satisfies

$$\mathbb{E}\left(\sup_{t\in\bar{\mathbb{N}}} |X_t^{i,I}|\right) < \infty.$$

- Main assumption (**A**)

$$\forall\,(i,j) \in \{1, \cdots N\},\ \ X_t^{i,\{i,j\}} \leq X_t^{i,\{j\}},\ \mathbb{P}\text{-a.s.}$$

Counter examples of non existence of (ϵ or true) NEP

- **Example 1 :** Deterministic setting with 2 players s.t.

$$\forall (i,j) \in \{1,2\},\ X_n^{i,\{j\}} = 0, X_n^{1,\{1,2\}} = 1 \text{ and } X_n^{2,\{1,2\}} = -1,$$

(A) not satisfied since $X_n^{1,\{1,2\}} = 1 > 0 = X_n^{1,\{2\}}, \ \forall n \in \bar{\mathbb{N}}$.
Hence $J_1(t_1,t_2) = -J_2(t_1,t_2) = 1_{(t_1=t_2)} \Rightarrow$ (the Game cannot have a ε-**N**EP for $\varepsilon < 1$.)

- **Example 2 :** For any $n \in \bar{\mathbb{N}}$ and any $i,j \in \{1,2\}$

$$X_n^{i,\{j\}} = \frac{n}{n+1}, X_n^{i,\{1,2\}} = 0 \quad (X_\infty^{i,\{j\}} = 1).$$

(A) is verified and for any $n,\ m \in \bar{\mathbb{N}}$,
$J_i(n,m) = \frac{n \wedge m}{n \wedge m + 1} 1_{\{n \neq m\}}$. However this game has no 0-**N**EP.

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The Snell enveloppe of processes : main properties

Let $U = (U_t)_{t\in\bar{\mathbb{N}}}$ be an $\mathbb{F}$-adapted $\mathbb{R}$-valued process such that $\mathbb{E}\left(\sup_{t\in\bar{\mathbb{N}}} |U_t|\right) < \infty$. For any $\mathbb{F}$-stopping time θ, let us define

$$Z(\theta) = \text{ess sup}_{\tau\in\mathcal{T}_\theta}\mathbb{E}[U_\tau|\mathcal{F}_\theta]\,(\text{and then } Z(\infty) = U_\infty). \qquad (1)$$

1. $(Z_n)_{n\geq 0}$ is a $\mathbb{F}$-supermartingale,
2. for $\varepsilon > 0$, the stopping time

$$\tau^* = \min\{s \geq 0, \quad Z_s \leq U_s + \varepsilon\},$$

is ε-optimal, *i.e.*,

$$\forall\tau \in \mathcal{T}_0 \quad \mathbb{E}[U_\tau] \leq \mathbb{E}[U_{\tau^*}] + \varepsilon, \mathbb{P}\text{-a.s.} \qquad (2)$$

Second part : Solving the non zero-sum N players Dynkin game

2.1 Presentation of the approximating schemes of stopping times

2.2 Convergence of sequences and characterization of their limit

Main steps of the algorithm

Initialization : setting $\tau_1 = \tau_2 = \cdots \tau_N = \infty$.

For $n \geq N+1$,

(a) $\theta_n = \min\{\tau_{n-1}, \cdots \tau_{n-N+1}\}$,

(b) $I^n := \{i_l \in \mathcal{J},\ n-N+1 \leq l \leq n-1 \text{ and } \tau_l = \theta_n\}$

(c) $U_t^n = X_t^{i_n,\{i_n\}} \mathbf{1}_{\{t<\theta_n\}} + Y^n \mathbf{1}_{\{t \geq \theta_n\}}, \ \forall\, t \in \bar{\mathbb{N}}$, with :

$Y^n = \left(X_{\theta_n}^{i_n, I^n \cup \{i_n\}} \vee X_{\theta_n}^{i_n, I^n}\right) \mathbf{1}_{\theta_n < \infty} + X_\infty^{i_n, \mathcal{J}} \mathbf{1}_{\theta_n = \infty}$

(d) $W_t^n = \text{ess sup}_{\tau \geq t} \mathbb{E}\left(U_\tau^n | \mathcal{F}_t\right)$,

(e) $\mu_n = \min\left(s \in \bar{\mathbb{N}},\ W_s^n \leq U_s^n + \epsilon\right)$,

(f) $\tau_n = \left(\mu_n \wedge \tau_{n-N}\right) \mathbf{1}_{\mu_n \wedge \tau_{n-N} < \theta_n} + \tau_{n-N} \mathbf{1}_{\mu_n \wedge \tau_{n-N} \geq \theta_n}$.

The main result : Construction of the ϵ-Nash equilibrium point

For each $i \in \{1, \cdots, N\}$, $(\tau_{Nq+i})_q$ and $(\Theta_{Nq+i})_q$ are $\searrow$ and convergent.

Considering both the N tuples (T_i^*) and (R_i^*) s.t

$$T_i^* := \lim \searrow \tau_{Nq+i} \text{ and } R_i^* := \lim \searrow \theta_{Nq+i},$$

- Let $R_i^* := \min(T_j^*, \, j \neq i) = \lim \searrow \theta_{Nq+i}$, and $R^* = T_i^* \wedge R_i^*$, $I_i^* = \{j \in \mathcal{J}, \ \text{s.t.} T_j^* = R^*\}$ consists in at most one index.
- $(T_1^*, \cdots T_i^*, \cdots, T_N^*)$ is an ϵ NEP for the NZSDG.

First auxiliary result

For each i_n, $(\tau_{Nq+i_n})_q$ satisfies

(i) $\mu_{Nq+i_n} \leq \tau_{N(q-1)+i_n}$ implying
for any q, $q \geq 1$,
$\tau_{Nq+i_n} = \mu_{Nq+i_n} \mathbf{1}_{\{\mu_{Nq+i_n} < \theta_{Nq+i_n}\}} + \tau_{N(q-1)+i_n} \mathbf{1}_{\{\mu_{Nq+i_n} \geq \theta_{Nq+i_n}\}}$

(ii) (τ_{Nq+i_n}) is decreasing (in $\bar{\mathbb{N}}$) :
$\Rightarrow$ converging + stationnary.

(iii) Since : $\mu_m = \tau_m \wedge \theta_m$, $(\mu_{Nq+i_n})_{q \geq 0}$ is decreasing and convergent.

Hence, one can define (T_i^*) and (R_i^*) as follows

$$T_i^* = \lim_q \searrow \tau_{Nq+i} \text{ and } R_i^* = \lim_q \searrow \theta_{Nq+i}.$$

First auxiliary result

Main steps of the proof of $\mu_n \leq \tau_{n-N}$

(i) by contradiction, set $m = \min\{n,\ \mathbb{P}\big(\mu_{n+N} > \tau_n\big) > 0\}$,
$m \geq N+1$,
and let define : $\Theta := \{\omega, \tau_m(\omega) < \mu_{m+N}(\omega)\}$ $(\Rightarrow\ \mathbb{P}(\Theta) > 0\,!)$

(ii) Prove :

$$\forall\, j \in \{1, \cdots, N-1\},\ \tau_{m+N-j} = \tau_{m-j},$$

and conclude that $\tau_m < \theta^{m+N} = \theta^m$ and hence

$$(\tau_m = \mu_m < \mu_{m+N})\ \ (1)$$

First auxiliary result

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(iii) Since : $\theta^m = \theta^{m+N} \Rightarrow \left(U^m = U^{m+N}\right)$ on the set Θ, one deduces :
(a) $\mathbf{1}_\Theta(W^{m+N}_{\mu_m}) \leq \mathbf{1}_\Theta(U_{\mu_m} + \epsilon)$,
(b) by the definition of μ_{m+N}, one obtains $\mu_{m+N} \leq \mu_m$ (2).

(iv) (1) and (2) are contradictory hence the result.

Second auxiliary result

We prove : $\forall\ m \geq N+1, \quad \mathbb{P}\big(\tau_m = \theta_m < \infty\big) = 0.$
taking $m = Nq + i$: $\mathbb{P}\big(\{\tau_{Nq+i} = \theta_{Nq+i}\}\big) = 0 \ \Rightarrow$
$\{T_i^* = R_i^*\}$ is null event !
Set $\Omega_m^{'} := \{\omega,\ \tau_m(\omega) = \theta_m(\omega) < \infty\}$

- Step 1 : Establish by Backward induction

$$\forall\ j \in \{m-N+1, \cdots, m-1\} \qquad \tau_j \leq \theta_m \text{ and } \tau_j = \tau_{j-N}.$$

- Step 2 : Deduce that $\Omega_m^{'} \subset \Omega_{m-N}^{'} \cdots \subset \Omega_{m-Nq}$
  Since $\exists\ q$ s.t. $m - Nq \in \{1, N\}$, then $\tau_{m-Nq} = \infty$ implying $\mathbb{P}(\Omega_m^{'}) = 0$

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The main result : Construction of the ϵ Nash equilibrium point

Key ingredients of the characterization of (T_i^*) as an ϵ-NEP

• Let $(\beta_n)_{n\geq 1}$: nondecreasing sequence of stopping times converging to β.

For any $i \in \{1, \cdots, N\}$, we have

$$\lim_n \mathbb{E} U^{Nn+N+i}_{\beta_n \wedge \theta_{Nn+N+i}} = \begin{aligned} &\mathbb{E}[J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, \beta, T_{i+1}^*, \cdots, T_N^*)] + \\ &\mathbb{E}\big(X^{i,I_i^*}_{R_i^*} - X^{i,I_i^* \cup \{i\}}_{R_i^*}\big)^+ 1_{\{R_i^* = \beta < \infty\}} + \\ &\mathbb{E}\big(X^{i,I_i^*}_{R_i^*} - X^{i,I_i^* \cup \{i\}}_{R_i^*}\big)^- 1_{\{R_i^* < \beta\}}. \end{aligned} \qquad (3)$$

The main result : Construction of the ϵ Nash equilibrium point

Key ingredients of the characterization : Using two times previous lemma with $\beta_n = \theta$ for the LHS, $\beta_n = \tau_{Nn+N+i}$ for the RHS + properties of τ_{Nn+N+i} as an ϵ optimal time (n fixed) :

For any $i \in \mathcal{J}$ and $\theta \in \mathcal{T}_0$,

$$\begin{array}{l}\mathbb{E}[J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, \theta, T_{i+1}^*, \cdots, T_N^*)]+\\ \mathbb{E}(X_{R_i^*}^{i,I_i^*} - X_{R_i^*}^{i,I_i^*\cup\{i\}})^+ 1_{\{R_i^*=\theta<\infty\}}+\\ \mathbb{E}(X_{R_i^*}^{i,I_i^*} - X_{R_i^*}^{i,I_i^*\cup\{i\}})^- 1_{\{R_i^*<\theta\}}\\ \\ \leq\\ \varepsilon + \mathbb{E}[J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, T_i^*, T_{i+1}^*, \cdots, T_N^*)]+\\ \mathbb{E}\left(X_{R_i^*}^{i,I_i^*} - X_{R_i^*}^{i,I_i^*\cup\{i\}}\right)^+ 1_{\{R_i^*=T_i^*<\infty\}}+\\ \mathbb{E}(X_{R_i^*}^{i,I_i^*} - X_{R_i^*}^{i,I_i^*\cup\{i\}})^- 1_{\{R_i^*<T_i^*\}}.\end{array} \quad (4)$$

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For any $i \in \mathcal{J}$ and $\theta \in \mathcal{T}_0$,

$$\begin{aligned}
&\mathbb{E}[J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, \theta, T_{i+1}^*, \cdots, T_N^*)]+ \\
&\mathbb{E}\left(X^{i,I_i^*}_{R_i^*} - X^{i,I_i^*\cup\{i\}}_{R_i^*}\right)^+ 1_{\{R_i^*=\theta<\infty\}}+ \\
&\mathbb{E}\left(X^{i,I_i^*}_{R_i^*} - X^{i,I_i^*\cup\{i\}}_{R_i^*}\right)^- 1_{\{R_i^*<\theta\}} \\
&\leq \\
&\varepsilon + \mathbb{E}[J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, T_i^*, T_{i+1}^*, \cdots, T_N^*)]+ \\
&\mathbb{E}\left(X^{i,I_i^*}_{R_i^*} - X^{i,I_i^*\cup\{i\}}_{R_i^*}\right)^+ 1_{\{R_i^*=T_i^*<\infty\}}+ \\
&\mathbb{E}\left(X^{i,I_i^*}_{R_i^*} - X^{i,I_i^*\cup\{i\}}_{R_i^*}\right)^- 1_{\{R_i^*<T_i^*\}}.
\end{aligned} \tag{4}$$

The main result : key auxiliary results

(1) Recall $R_i^* = \min(T_j^*,\ j \neq i)$ then

$$\mathbb{P}\,(T_i^* = R_i^* < \infty) = 0$$

$\Rightarrow I_i^* = \{j\}$.

(2)

$$\mathbb{E}\left((X_{R_i^*}^{i,I_i^*} - X_{R_i^*}^{i,I_i^* \cup \{i\}})^- 1_{\{R_i^* < T_i^*\} \leq 0}, \right)$$

thanks to Assumption (A) and previous item (1).

Third part : Conclusion

1. Under Assumption A + integrability conditions :
   $\Rightarrow$ existence of an ϵ NEP.
2. Previously in Hamadene-Hassani and assuming :
   $\overline{\lim}_{t\to\infty} X_t^I \leq X_\infty^I$,
   $\Rightarrow$ existence of a true NEP.
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   Seems non trivial (stationnarity of converging sequences deeply used)

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