On the value of a zero-sum switching game

S.Hamadène

University of Maine (Le Mans, France).
(jww B.Djehiche, M.A.Morlais and X.Zhao.)

French Symposium on Game Theory and Applications: May 26-30, 2015, University Paris-Diderot.

0. Introduction. The control case.

Let:

(i) $(\Omega, \mathbb{F}, (F_t)_{t\leq T}, \mathbf{P}, (B_t)_{t\leq T})$ be a stochastic basis ; $(B_t)_{t\leq T}$ is a BM .

(ii) For $(t,x) \in [0,T] \times \mathbb{R}^k$, $X^{t,x}$ verifies:

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s,\ \ s \in [t,T]\ ;\ \ X_s^{t,x} = x \text{ if } s \leq T$$

An stochastic switching problem is a control where strategies $\delta := (\tau_n, \xi_n)_{n\geq 0}$ have two components:

(i) $(\tau_n)_{n\geq 0}$ is an increasing sequence of stopping times with values in $[0, T]$;

(ii) ξ_n is a random variable F_{τ_n}-measurable with values in $\Gamma^1 := \{1, ..., m_1\}$;

(iii) When the system is in mode/state $i \in \Gamma^1$ at times s the instantaneous utility during ds is $f_i(s, X_s^{t,x})ds$;

(iv) Switching the system from mode i to mode $j \neq i$ at time s generates a cost

$$\underline{g}_{ij}(s, X_s^{t,x})$$

(v) When a strategy δ is implemented the payoff is:

$$J(\delta) := \mathbf{E}[h^{a_T}(X_T^{t,x}) + \int_0^T f_{a_r}(r, X_r^{t,x})dr - \sum_{n \geq 1} \underline{g}_{\xi_{n-1}\xi_n}(\tau_n, X_{\tau_n}^{t,x}) 1_{\{\tau_n < T\}}]$$

where $(a_s)_{s \leq T}$ is the indicator of the state of the system along with time and connected with δ.

The HJB system of PDEs with obstacles associated with this switching problem is: for $i \in \Gamma^1$,

$$\begin{cases} \min \Big[v^i(t,x) - \max_{j \neq i}\{v^j(t,x) - \underline{g}_{ij}(t,x)\}\,; \\ \qquad\qquad\qquad -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) - f_i(t,x) \Big] = 0, \\ v^i(T,x) = h^i(x) \end{cases} \tag{1}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}\varphi(t,x) := b(t,x)D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)D^2_{xx}\varphi(t,x)],$$

For $i \in \Gamma^1$, let

$$u^i(t,x) := \sup_{\delta \in \mathcal{A}_t^i} \mathbf{E}[h^{a_T}(X_T^{t,x}) + \int_t^T f_{a_r}(r, X_r^{t,x})dr - \sum_{n\geq 1} \underline{g}_{\xi_{n-1}\xi_n}(\tau_n, X_{\tau_n}^{t,x}) 1_{\{\tau_n < T\}}].$$

Theorem (H.-Morlais, '12)

If:

(i) $\underline{g}_{ij}$ *are non-negative and satisfy the no free loop property i.e. for any* $i_1, ..., i_k$ *such that* $i_1 = i_k$ *and* $|\{i_1, ..., i_k\}| = k - 1$ *then for any* (t, x),

$$\underline{g}_{i_1 i_2}(t,x) + ... + \underline{g}_{i_{k-1} i_k}(t,x) > 0\,;$$

(ii) $\forall i \in \Gamma^1$

$$h^i(x) \geq \max_{j \neq i}\{h^j(x) - \underline{g}_{ij}(T,x)\}\,;$$

(iii) The functions are continuous with polynomial growth at ∞.

Then the functions $(u^i(t,x))_{i\in\Gamma^1}$ *are continuous with polynomial growth at* ∞ *and are unique viscosity solution of the HJB system (2).*

1. The Zero-sum switching game

(i) There are two players P1 and P2 who act on the system. The modes for P1 (resp. P2) are Γ^1 (rep. $\Gamma^2 := \{1, ..., m_2\}$) who chooses i (resp. j). The working modes of the system are $(i,j) \in \Gamma^1 \times \Gamma^2$;

(ii) f^{ij} and h^{ij} are related to the instantaneous and terminal payoffs respectively which are profits (resp. costs) for P_1 (resp. P_2) ;

(iii) $\underline{g}_{ij}(t,x)$ (resp. $\overline{g}_{ij}(t,x)$) is a function related to the switching costs for P_1 (resp. P_2) ;

(iv) Similarly, a strategy of switching for P_2 consists of two components: the times when the decision to switch is made and to which state the system is switched. It has the form $\nu = (\sigma_n, \zeta_n)_{n\geq 0})$ where (σ_n is an $\mathbb{F}$-s.t.and $\zeta_n \in \Gamma^2$, $\mathbb{F}_{\sigma_n}$-meas.).

(v) When P_1 (resp. P_2) implements the strategy δ (resp. ν) there is a payoff $J(\delta,\nu)$ (depending on $f^{ij}, h^{ij}, \underline{g}_{ij}$ and $\bar{g}_{ij}$) which is a profit for P_1 and a cost for P_2 and given by:

$$J(\delta,\nu) := \mathbf{E}[h^{a_T b_T}(X_T^{t,x}) + \int_0^T f(r, X_r^{t,x}, a_r, b_r)dr - A_T^\delta + B_T^\nu]$$

where for $t \leq T$

$$A_t^\delta = \sum_{n\geq 1} \underline{g}_{\xi_{n-1}\xi_n}(\tau_n, X_{\tau_n}^{t,x})1_{\{\tau_n<t\}} \text{ and } B_t^\nu = \sum_{n\geq 1} \bar{g}_{\zeta_{n-1}\zeta_n}(\sigma_n, X_{\sigma_n}^{t,x})1_{\{\sigma_n<t\}};$$

a_s (resp. b_s) is the state choosen by P1 (resp. P2) at time s.

We are then interested in the quantities

$$\inf_{\nu} \sup_{\delta} J(\delta, \nu) \text{ and } \sup_{\delta} \inf_{\mu} J(\delta, \nu).$$

The game has a value when they are equal.

3. HJB Systems of ZS Switching Game

- For $(i,j) \in \Gamma^1 \times \Gamma^2$

$$(\Gamma^1)^{-i} := \Gamma^1 - \{i\} \text{ and } (\Gamma^2)^{-j} := \Gamma^2 - \{j\}.$$

- $\forall (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} \min\Big[(v^{ij} - L^{ij}[\vec{v}])(t,x); \\ \quad \max\Big\{(v^{ij} - U^{ij}[\vec{v}])(t,x); -\partial_t v^{ij}(t,x) - \mathcal{L}v^{ij}(t,x) - \\ \qquad\qquad f^{ij}(t,x)\Big\}\Big] = 0, \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{2}$$

where:

- For $i \in \Gamma^1$ and $j \in \Gamma^2$,

$$L^{ij}[\vec{v}](t,x) := \max_{k \in (\Gamma^1)^{-i}} (v^{kj}(t,x) - \underline{g}_{ik}(t,x)\} \text{ (lower obstacle)}$$

$$U^{ij}[\vec{v}](t,x) := \min_{l \in (\Gamma^2)^{-j}} (v^{il}(t,x) + \bar{g}_{jl}(t,x)) \text{ (upper obstacle)};$$

- The unknowns are the functions $(v^{ij}(t,x))_{(i,j) \in \Gamma^1 \times \Gamma^2}$

Dually one can consider: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$,

$$
\begin{cases}
\max \Big[\big(v^{ij} - U^{ij}[\vec{v}]\big)(t,x); \\
\quad \min \Big\{ \big(v^{ij} - L^{ij}[\vec{v}]\big)(t,x); -\partial_t v^{i,j}(t,x) - \mathcal{L} v^{ij}(t,x) - \\
\qquad\qquad f^{ij}(t,x) \Big\} \Big] = 0, \\
v^{ij}(T,x) = h^{ij}(x).
\end{cases}
\tag{3}
$$

3.1. Existence and uniqueness of a solution for system (2)

Assumptions

(H1) Each function f^{ij} is continuous and has polynomial growth ;

(H2) The functions h^{ij} are continuous, of polynomial growth (PG) and satisfy: $\forall (i,j)$,

$$\max_{k\in(\Gamma^1)^{-i}} \left(h^{kj}(x) - \underline{g}_{ik}(T,x)\right) \leq h^{ij}(x) \leq \min_{l\in(\Gamma^2)^{-j}} \left(h^{il}(x) + \bar{g}_{jl}(T,x)\right).$$

(H3) The no free loop property

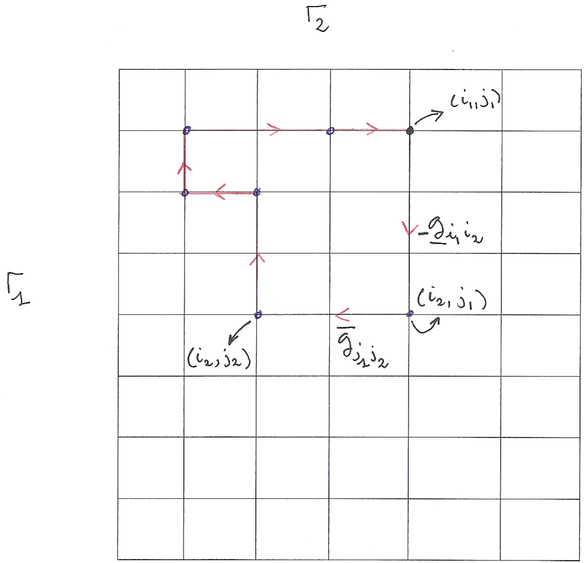
The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t, x) and satisfy:

For any loop in $\Gamma^1 \times \Gamma^2$, i.e., pairs $(i_1, j_1), \ldots, (i_N, j_N)$ such that $(i_N, j_N) = (i_1, j_1)$, $\text{card}\{(i_1, j_1), \ldots, (i_N, j_N)\} = N - 1$ and $\forall\ q = 1, \ldots, N-1$, either $i_{q+1} = i_q$ or $j_{q+1} = j_q$, we have:

$$\sum_{q=1,N-1} \varphi_{i_q j_q}(t, x) \neq 0, \tag{4}$$

where, $\forall\ \ q = 1, \ldots, N-1$,

$$\varphi_{i_q j_q}(t, x) = -\underline{g}_{i_q i_{q+1}}(t, x) 1\!\!1_{i_q \neq i_{q+1}} + \bar{g}_{j_q j_{q+1}}(t, x) 1\!\!1_{j_q \neq j_{q+1}}.$$



Γ_2
Γ_1
(i_1, j_1)
$-g_{i_1 i_2}$
(i_2, j_1)
(i_2, j_2)
$\bar{g}_{j_1 j_2}$

Theorem (Djehiche, H., Morlais, '15)

Under (H1)-(H3), the system of PDEs with bilateral interconnected obstacles (2) has a unique solution (in viscosity sense) in the classe of continuous functions with polynomial growth.

The proof is based on Perron's method.

Step 1: Comparison and uniqueness.

If $(u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $(w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is a subsolution (resp. supersolution) of system (2) with polynomial growth then

$$\forall (i,j)\in\Gamma^1\times\Gamma^2,\ u^{ij}\le w^{ij}.$$

As a consequence the system has at most one solution with polynomial growth which is then continuous. The proof is based on CIL's Lemma.

Step 2 : Existence of a subsolution.

- The penalization schemes.

(i) Let $(\bar{v}^{i,j,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the solution of: $\forall\ (i,j)\in\Gamma^1\times\Gamma^2$,

$$
\begin{cases}
\min\Big\{\bar{v}^{ij,m}(t,x) - \max_{k\in(\Gamma^1)^{-i}}(\bar{v}^{kj,m}(t,x) - \underline{g}_{ik}(t,x)); \\
\qquad -\partial_t \bar{v}^{ij,m}(t,x) - \mathcal{L}\bar{v}^{ij,m}(t,x) \\
\qquad\qquad -\bar{f}^{ij,m}(t,x,(\bar{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2})\Big\} = 0; \\
\bar{v}^{ij,m}(T,x) = h^{ij}(x).
\end{cases}
\tag{5}
$$

where

$$\bar{f}^{ij,m}(t,x,(\bar{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}) :=$$

$$f^{ij}(t,x) - m(\underline{v}^{ij,m}(t,x) - \min_{l\in(\Gamma^2)^{-j}}\{\bar{v}^{il,m}(t,x) + \bar{g}_{jl}(t,x)\})^+$$

Then

$$\bar{v}^{ij,m} \geq \bar{v}^{ij,m+1}.$$

(ii) Let $(\underline{v}^{i,j,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the unique solution of the following system of VIs with one interconnected obstacle: $\forall\ (i,j)\in\Gamma^1\times\Gamma^2$

$$\begin{cases} \max\Big\{\underline{v}^{ij,m}(t,x)-\min_{l\in(\Gamma^2)^{-j}}(\underline{v}^{il,m}(t,x)+\bar{g}_{jl}(t,x)); \\ -\partial_t\underline{v}^{ij,m}(t,x)-\mathcal{L}\underline{v}^{ij,m}(t,x) \\ \qquad -\underline{f}^{ij,m}(t,x,(\underline{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2})\Big\}=0, \\ \underline{v}^{ij,m}(T,x)=h^{ij}(x) \end{cases} \tag{6}$$

where

$$\underline{f}^{ij,m}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}) =$$

$$f^{ij}(s,x)+m\big(y^{ij}-\max_{k\in(\Gamma^1)^{-i}}\{y^{kj}-\underline{g}_{ik}(s,x)\}\big)^-.$$

Once more comparison implies that:

$$\underline{v}^{ij,m} \leq \underline{v}^{ij,m+1}$$

and

$$\underline{v}^{ij,m} \leq \bar{v}^{ij,m}.$$

Let:

$$\bar{v}^{ij}(t,x) := \lim_{m\to\infty} \bar{v}^{ij,m}(t,x) \quad \text{and} \quad \underline{v}^{ij}(t,x) := \lim_{n\to\infty} \underline{v}^{ij,n}(t,x).$$

Lemma

$(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is a viscosity subsolution of (2)*

Step 3 : Existence of a supersolution.

$\bar{v}^{ij,m_0}$ is the unique viscosity solution of

$$\begin{cases} \min\Big[\vartheta(t,x) - \max_{k\in(\Gamma^1)^{-i}}\{\bar{v}^{kj,m_0}(t,x) - \underline{g}_{ik}(t,x)\}; \\ \max\Big\{\vartheta(t,x) - \bar{v}^{ij,m_0}(t,x) \vee \min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) + \bar{g}_{jl}(t,x)); \\ \qquad -\partial_t\vartheta(t,x) - \mathcal{L}\vartheta(t,x)) - f^{ij}(t,x)\Big\}\Big] = 0\,; \\ \vartheta(T,x) = h^{ij}(x). \end{cases}$$

Then $(\bar{v}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity supersolution of system (2) because $a - (a \vee b) \leq a - b$.

Step 4 : Perron's method

Let m_0 be fixed and let $\mathcal{U}_{m_0}$ defined as follows:

$$\mathcal{U}_{m_0} := \{\vec{u} = (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2} \text{ s.t. } \vec{u} \text{ is a subsolution of (2)} \\ \text{and } \forall (i,j) \in \Gamma^1 \times \Gamma^2,\ \bar{v}^{i,j} \leq u^{i,j} \leq \bar{v}^{ij,m_0}\}.$$

$\mathcal{U}_{m_0}$ is not empty since it contains $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$. For (t,x) and $(i,j) \in \Gamma^1 \times \Gamma^2$, let us set:

$$^{m_0}v^{ij}(t,x) = \sup\{u^{ij}(t,x),\ (u^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} \in \mathcal{U}_{m_0}\}.$$

Theorem (Djehiche-H.-Morlais, '15)

The family $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *does not depend on* m_0 *and is the unique continuous viscosity solution in the class of PG functions of system (2). Moreover for any* $(i,j) \in \Gamma^1 \times \Gamma^2$, ${}^{m_0}v^{ij} = \bar{v}^{ij}$.

The last equality comes from

$$v^{ij,m_0} \geq {}^{m_0}v^{ij} \geq \bar{v}^{ij}.$$

Corollary

The limit of the increasing scheme $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is continuous and is the unique viscosity solution (in the class PG) for the following system: For any* $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \max\Big\{\big(\underline{v}^{ij}-U^{ij}[\underline{\vec{v}}]\big)(t,x);\ \ \min\Big[\big(\underline{v}^{ij}(t,x)-L^{ij}[\underline{\vec{v}}]\big)(t,x), \\ \quad -\partial_t\underline{v}^{ij}(t,x)-\mathcal{L}\underline{v}^{ij}(t,x) \\ \qquad -f^{ij}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\underline{v}^{ij}(t,x))\Big]\Big\}=0; \\ \underline{v}^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{7}$$

4. Link with zerosum switching game

Question 1: $\bar{v}^{ij} = \underline{v}^{ij}$?

Proposition

If for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$L^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}] \leq U^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}] \quad (8)$$

then for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $\bar{v}^{ij} = \underline{v}^{ij}$.

Proof: Under (8), for any (i,j), $\bar{v}^{i,j}$ is also a solution for the max-min problem and by uniqueness, $\bar{v}^{ij} = \underline{v}^{ij}$. ■

(H4): On the condition (8).

- For any k, l, $\bar{g}_{kl}(t,x) \in \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^k)$;
- For any (i,j),

$$|f^{ij}(t,x,\vec{y},z)| \leq C(1+|x|^p).$$

Proposition

Under (H1)-(H4), inequality (8) holds true.

Proof:

- First note that

$$\bar{g}_{kl}(s, X_s^{t,x}) = \bar{g}_{kl}(t,x) + \int_t^s \bar{u}_r dr - \int_t^s \bar{v}_r dB_r, \ \ t \leq s \leq T,$$

with $\bar{u}$, $\bar{v}$ $dt \otimes d\mathbf{P}$-square integrable.

• We then use systems of reflected backward equations to show that: for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$m^2 \mathbf{E}[\int_s^T \{(\bar{v}^{ij,m}(r, X_r^{t,x}) - \min_{q \neq j}\{\bar{v}^{iq,m}(r, X_r^{t,x}) + \bar{g}_{jq}(r, X_r^{t,x})\})^+\}^2 dr] \leq C. \tag{9}$$

• Finally the fact that $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity solution of (2) allows to conclude that

$$L^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}] \leq \bar{v}^{ij} \leq U^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}].$$

Actually

$$\bar{v}^{ij}(t,x) \geq L^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}](t,x).$$

(i) If $\bar{v}^{ij}(t,x) > L^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}](t,x)$ then by the subsolution property $\bar{v}^{ij}(t,x) \leq U^{ij}[(\bar{v}^{kl})_{(k,l)}](t,x)$;

(ii) If $\bar{v}^{ij}(t,x) = L^{ij}[(\bar{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}](t,x)$ and $\bar{v}^{ij}(t,x) > U^{ij}[(\bar{v}^{kl})_{(k,l)}](t,x)$ then this leads to the explosion of (9) which is contradictory.

Corollary

Under (H1)-(H4), systems (2) and (3) have the same solution.

Question 2 : Connection with the zero-sum switching game.

Let δ (resp. ν) a strategy of P_1 (resp. P_2). The payoff for the players is given by:

$$J_t^{ij}(\delta,\nu) := \mathbf{E}[h^{a_T,a_T}(T,X_T^{t,x}) + \int_t^T f^{a_r,b_r}(r,X_r^{t,x})dr - A_T^{\delta} + B_T^{\nu}]$$

where for $s \in [t, T]$,

$$A_s^{\delta} = \sum_{n\geq 1} g_{\xi_{n-1},\xi_n}(\sigma_n, X_{\sigma_n}^{t,x})1_{[\sigma_n \leq s]} \text{ and}$$

$$B_s^{\nu} = \sum_{n\geq 1} \bar{g}_{\zeta_{n-1},\zeta_n}(\tau_n, X_{\tau_n}^{t,x})1_{[\tau_n \leq s]}.$$

Theorem

Under (H1)-(H6),

$$\bar{v}^{ij}(t,x) = \sup_{\delta}\inf_{\nu} J_t^{ij}(\delta,\nu) = \inf_{\nu}\sup_{\delta} J_t^{ij}(\delta,\nu).$$

Thanks for your attention.