A probabilistic representation for continuous-time games with incomplete information on both sides.

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Plan

1. Zerosum continuous time games with incomplete information.
2. Lack of information on one side
3. A continuous version of the splitting game.

Zerosum continuous time games with incomplete information.

- Control spaces : U, V compact, metric.
- Family of cost functions : For $I, J \in \mathbb{N}^*$,

$$J_{ij}(t, u_., v_.) = \int_t^T \ell_{ij}(s, u_s, v_s)ds, \ (i,j) \in \{1, \dots, I\} \times \{1, \dots, J\}.$$

(u_s) (resp. (v_s)) U (resp. V valued), measurable

ℓ_{ij} Lipschitz in t, uniformly in u, v.

- Probability measures : $p = (p_1, \dots, p_I) \in \Delta(I)$, where $\Delta(I)$ is the set of probability measures on $\{1, \dots, I\}$ (resp. $q \in \Delta(J)$ probability on $\{1, \dots, J\}$).

$J_{ij}(t, u_\cdot, v_\cdot) = \int_t^T \ell_{ij}(s, u_s, v_s)ds$,
$(p, q) \in \Delta(I) \times \Delta(J)$.

The game is played in two steps:

- At initial time t the indexes (i, j) are chosen at random according to the probability $p \otimes q$
- i is communicated to Player 1 only, j only to Player 2.
- Then
    - Player 1 plays (u_s), tries to minimize the payoff $J_{ij}(t, u_\cdot, v_\cdot)$,
    - Player 2 plays (v_s), tries to maximize this payoff,
- both players observe the action of their opponent.

Values.

$J_{ij}(t,u,v) = \int_t^T \ell_{ij}(s,u_s,v_s)ds$, $(p,q) \in \Delta(I) \times \Delta(J)$

Upper value: $V^+(t,p,q) := \inf_{(\alpha_i)\in\mathcal{A}_r(t)^I} \sup_{\beta_j\in\mathcal{B}_r(t)^J} \left(\sum_{i,j} p_i q_j E_{\alpha_i,\beta_j}[J_{ij}(t,u.,v.)] \right)$,

Lower value: $V^-(t,p,q) := \sup_{\beta_j\in\mathcal{B}_r(t)^J} \inf_{(\alpha_i)\in\mathcal{A}_r(t)^I} \left(\sum_{i,j} p_i q_j E_{\alpha_i,\beta_j}[J_{ij}(t,u.,v.)] \right)$,

where $\mathcal{A}_r(t)$ (resp. $\mathcal{B}_r(t)$) is the set of strategies for Player 1
$\alpha : (v_s) \to (u_s)$ which are

- non anticipative with delay
- random,

$\mathcal{B}_r(t)$ the strategies for Player 2,
and $P_{\alpha,\beta}$ the probability induced on $(u.,v.)$ by (α,β).

$J_{ij}(t, u_\cdot, v_\cdot) = E[\int_t^T \ell_{ij}(s, u_s, v_s)ds]$,
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Theorem (Cardaliaguet 2007)

Under Isaacs' condition, the game has a value: $V^+ = V^- := V$ *which is the unique Lipschitz continuous viscosity solution of:*

$$\begin{cases} \max\{\min\{\frac{\partial V}{\partial t} + H\,;\, \lambda_{\min}(D_p^2 V)\}\,;\, \lambda_{\max}(D_q^2 V)\} = 0, \\ V|_{t=T} = 0, \end{cases} \qquad (1)$$

where $\lambda_{min}(A)$ (resp. $\lambda_{max}(A)$) is the smallest (resp. largest) eigenvalue of A, and

$$H(t, p, q) = \min_u \max_v \sum_{i,j} p_i q_j \ell_{ij}(t, u, v).$$

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Lack of information on one side

$I \geq 1$ (and J=1), $\ell_i : [0, T] \times U \times V \to \mathbb{R}, i \in \{1, \dots, I\}$,
$V(t, p) = \inf_{(\alpha_i) \in \mathcal{A}_r^i} \sup_{\beta \in \mathcal{B}_r} \sum_i p_i E_{\alpha_i, \beta}[\int_t^T \ell_i(s, u_s, v_s) ds]$.

Theorem (Cardaliaguet R. 2009)

Under Isaac's condition,

$$V(t, p) = \min_{(p_s) \textit{ martingales } \to \Delta(I), p_t = p} E[\int_t^T H(s, p_s) ds],$$

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Interpretation: (p_s) martingales of believe for Player 2.
$\longrightarrow$ optimal strategy for the informed Player 1.

Generalizations: Grün 2012, Cardaliaguet R. 2012

Back to incomplete information on both sides.

$$V(t,p,q) = \inf_{(\alpha_i)\in\mathcal{A}_r^I} \sup_{\beta_j\in\mathcal{B}_r^J} \sum_{ij} p_i q_j E_{\alpha_i,\beta_j}[\int_t^T \ell_{ij}(s,u_s,v_s)ds].$$

Do we have some representation of type

$$V(t,p,q) = \max_{\beta} \min_{(p_s)\ \text{mart.}\to\Delta(I),p_t=p} E[\int_t^T H(s,p_s,\beta(p)_s)ds],$$

with $H(s,p,q) = \min_u \max_v \sum_{ij} p_i q_j \ell_{ij}(s,u,v)$
and $\beta : \{(p_s) \text{ martingale} \to \Delta(I)\} \to \{(q_s) \text{ martingale} \to \Delta(J)\}$?

Repeated games, Laraki 2001 : the splitting game.

A continuous version of the splitting game.

Let (B_s^1) and (B_s^2) be two independent Brownian motions with values in $\mathbb{R}^I$, resp. $\mathbb{R}^J$.
For $t \in [0, T]$,
let A_t be the set of $\mathcal{F}^{B^1,B^2}$-adapted $\mathbb{R}^{I\times I}$-valued processes,
(resp. B_t the set of $\mathcal{F}^{B^1,B^2}$-adapted $\mathbb{R}^{J\times J}$-valued processes).
For $(p, q) \in \Delta(I) \times \Delta(J)$, $(a_s, b_s) \in A_t \times B_t$,

$$\begin{array}{ll} p_s = p + \int_t^s a_r dB_r^1, & \\ q_s = q + \int_t^s b_r dB_r^2, & s \in [t, T], \end{array} \tag{2}$$

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For $(p, q) \in \Delta(I) \times \Delta(J)$, $(a_s, b_s) \in A_t \times B_t$,

$$\begin{aligned} p_s &= p + \int_t^s \Pi^I_{p_r} a_r dB^1_r, \\ q_s &= q + \int_t^s \Pi^J_{q_r} b_r dB^2_r, \; s \in [t, T], \end{aligned} \tag{2}$$

where, for all $w \in \mathbb{R}^I$, $\Pi^I_p w$ is the projection of w on the tangent space on p to $\Delta(I)$ (+ analogue definition for Π^J_q).

Theorem

- *Equations (2) have unique strong solutions (p_s) and (q_s),*
- *For all $s \in [t, T]$ P-a.s., $p_s \in \Delta(I)$ (resp. $q_s \in \Delta(J)$).*

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$p_s^a = p + \int_t^s \Pi_{p_r^a}^I a_r dB_r^1,$
$q_s^b = q + \int_t^s \Pi_{q_r^b}^J b_r dB_r^2, \; s \in [t, T],$

Value functions :

$$W^+(t,p,q) = \inf_\alpha \sup_\beta E[\int_t^T H(s, p_s^a, q_s^b)ds],$$
$$W^-(t,p,q) = \sup_\beta \inf_\alpha E[\int_t^T H(s, p_s^a, q_s^b)ds],$$

with α (resp. β) non anticipative strategies with delay : $B_t \to A_t$ (resp. $A_t \to B_t$).

Proposition

W^+ and W^- are Lipschitz in t, p, q, convex in p, concave in q.

$W^+(t,p,q) = \inf_\alpha \sup_\beta E[\int_t^T H(s,p_s^a,q_s^b)ds]$,
$W^-(t,p,q) = \sup_\beta \inf_\alpha E[\int_t^T H(s,p_s^a,q_s^b)ds]$.

Proposition

$W^+ = W^- := W$ is the unique Lipschitz continuous solution in viscosity sense of

$$(1) \quad \begin{cases} \max\{\min\{\frac{\partial V}{\partial t} + H\,;\, \lambda_{\min}(D_p^2 V)\}\,;\, \lambda_{\max}(D_q^2 V)\} = 0, \\ V|_{t=T} = 0, \end{cases}$$

Corollary

W coincides with the value of the continuous time, zerosum game with incomplete information on both sides : $W = V$.

Possible extensions

- More general compact convex sets C, D instead of $\Delta(I), \Delta(J)$.
- More general PDE with the same convexity constraints

  $$\max\{\min\{\frac{\partial V}{\partial t} + \mathcal{L}(V) + u\,;\, \lambda_{\min}(D_p^2 V)\}\,;\, \lambda_{\max}(D_q^2 V)\} = 0,$$

  Already appearing in models of continuous-time Markov games with incomplete information: see Cardaliaguet, R, Rosenberg, Vieille 2013 and Gensbittel 2013.
  (probably requires viability theory)
- PDE with different obstacles ?

Thank you for your attention!