Uniqueness of equilibrium

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Joint work with Thomas Pradeau

CERMICS, Optimisation et Systèmes

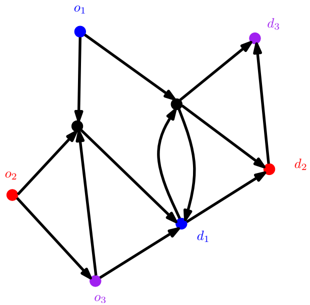
Model

$D = (V, A)$ a directed graph.

$\mathcal{L} \subseteq V^2$ a set of origin-destination pairs.

b^{od} = number of players going from o to d (the demand).

On each arc $a \in A$, there is a continuous cost $c_a(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$.



x_a = number of players choosing arc a.

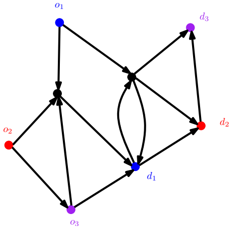
$\sum_{a \in P} c_a(x_a)$ = cost of a path P.

Equilibrium

x_a^{od} = number of players choosing arc a among those going from o to d.

$(x_a^{od})_{a \in A,(o,d) \in \mathcal{L}}$ is an equilibrium if

$(x_a^{od})_{a \in A} = o\text{-}d$ flow of value b^{od}	$(o,d) \in \mathcal{L}$
$x_a = \sum_{(o,d) \in \mathcal{L}} x_a^{od}$	$a \in A$
$\sum_{a \in P} c_a(x_a) \leq \sum_{a \in Q} c_a(x_a)$	$P, Q \in \mathcal{P}^{od}$, P is used, $(o,d) \in \mathcal{L}$

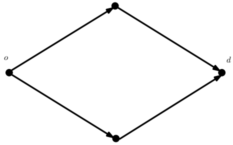


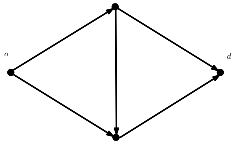
$\mathcal{P}^{od}$ = set of o-d paths

P used if $x_a^{od} > 0$ for all $a \in P$.

Practical interest

- This model = good approximation of what happens in practice
  - ⋆ used in transport engineering, telecoms,...
- Useful since the phenomenons are nonintuitive
  - ⋆ Braess paradox = opening a new road may increase all travel times
  - ⋆ paradox recovered by the model





Existence and uniqueness of the equilibrium

Theorem (Beckman, 1956)

An equilibrium always exists and it is "unique" when the cost functions $c_a(\cdot)$ are increasing.

"Unique" means there are unique x_a's solutions of the system

$$\begin{cases} (x_a^{od})_{a\in A} = o\text{-}d \text{ flow of value } b^{od} & (o,d)\in\mathcal{L} \\ x_a = \sum_{(o,d)\in\mathcal{L}} x_a^{od} & a\in A \\ \sum_{a\in P} c_a(x_a) \leq \sum_{a\in Q} c_a(x_a) & P,Q \in \mathcal{P}^{od}, \; P \text{ is used}, \; (o,d)\in\mathcal{L} \end{cases}$$

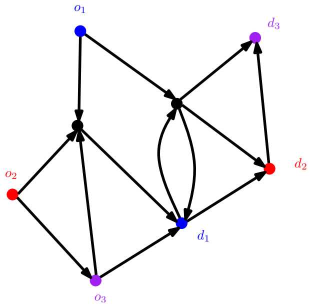
Model – multiclass case

$D = (V, A)$ a directed graph.

$\mathcal{L} \subseteq V^2$ a set of origin-destination pairs.

$b^{od,k}$ = number of class k players going from o to d (the demand).

On each arc $a \in A$ and for each class k, there is a continuous cost $c_a^k(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$.



x_a = number of players choosing arc a.

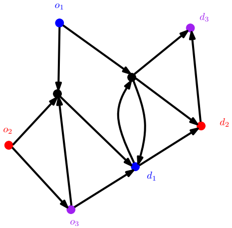
$\sum_{a\in P} c_a^k(x_a)$ = cost of path P experienced by class k.

Equilibrium

$x_a^{od,k}$ = number of class k players choosing arc a among those going from o to d.

$(x_a^{od,k})_{a\in A,(o,d)\in\mathcal{L},k\in K}$ is an equilibrium if

$(x_a^{od,k})_{a\in A} = o\text{-}d$ flow of value $b^{od,k}$	$(o,d)\in\mathcal{L}$, $k\in K$
$x_a = \sum_{(o,d)\in\mathcal{L},\,k\in K} x_a^{od,k}$	$a\in A$
$\sum_{a\in P} c_a^k(x_a) \leq \sum_{a\in Q} c_a^k(x_a)$	$P,Q\in\mathcal{P}^{od}$, P is used, $(o,d)\in\mathcal{L}$, $k\in K$



$\mathcal{P}^{od}$ = set of o-d paths

P used if $x_a^{od,k} > 0$ for all $a\in P$.

The uniqueness issue

Theorem (Schmeidler, 1973)

An equilibrium always exists in the multiclass setting.

There are examples with several equilibria, with several possible x_a's, while all $c_a^k(\cdot)$ are increasing: uniqueness of the equilibrium flows is not automatically ensured. (It contrasts with the monoclass case).

Challenge: Find necessary and/or sufficient conditions ensuring uniqueness.

Uniqueness: cost-based sufficient conditions

Proposition (Aashtiani, Magnanti, 1981)

If the players' cost functions are identical up to additive constants, then, for every two Nash equilibria, the flow on each arc in the network in the first equilibrium is equal to that in the second.

Uniqueness property

G = undirected graph, $\mathcal{L}$ = collection of o-d pairs.

$(G, \mathcal{L})$ has the uniqueness property (UP) if for any classes, demands $(b^{od,k})$, and increasing costs $(c_a^k(\cdot))$, the equilibrium flows are unique.
(on the digraph where each edge has been replaced by two opposite arcs)

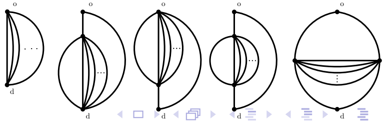
Theorem (Milchtaich, 2007)

There is only one o-d pair:

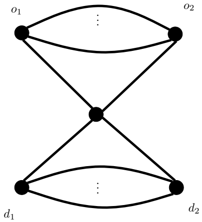
$$(G, \{(o,d)\}) \text{ has the UP} \quad \Longleftrightarrow \quad G \text{ is "nearly-parallel".}$$

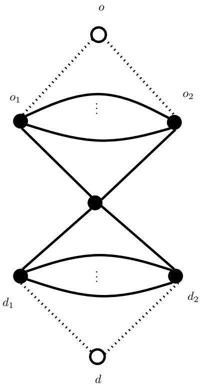
"Nearly-parallel" $=$

combination in series of



Uniqueness for general graphs using Milchtaich's theorem





- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

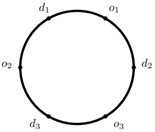
Augmented graph has uniqueness property $\Rightarrow$ Original graph has uniqueness property.

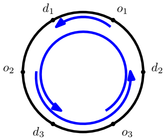
Uniqueness for general graphs using Milchtaich's theorem

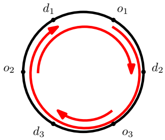
- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has not the uniqueness property $\Rightarrow$????

Uniqueness property on a cycle

$(G, \mathcal{L}) =$ 



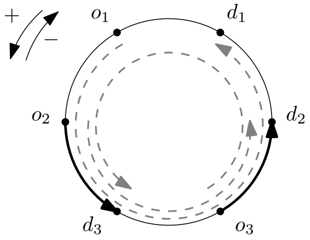


Theorem (M., Pradeau, 2014)

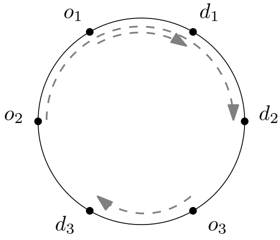
Assume that G is a cycle and let $\mathcal{L}$ be any collection of o-d pairs.

$(G, \mathcal{L})$ has the UP $\iff$ Each arc belongs to at most two o-d paths.

Example not having the uniqueness property



The positive paths.



The negative paths.

The arcs o_2d_3 and o_3d_2 are contained in three paths.

Proof strategy

Step 1.

Each arc of D belongs to at most two o-d paths $\Downarrow$ *The equilibrium flows are unique whatever are the classes* K, *increasing costs* $(c_a^k(\cdot))$, *and demands* $(b^{od,k})$

Step 2.

There is an arc of D belonging to at least three o-d paths $\Downarrow$ *There exist classes* K, *increasing costs* $(c_a^k(\cdot))$, *and demands* $(b^{od,k})$ *leading to two equilibria with distinct flows*

Proof by an explicit construction of costs and demands.

Step 1:

Each arc of D belongs to at most two o-d paths

$\Downarrow$

The equilibrium flows are unique whatever are the classes, costs, and demands

- Let $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ be two equilibria. Define $\Delta_{od} = x^{od}_{P^+} - \hat{x}^{od}_{P^+}$.
- Suppose $\Delta_{o_0d_0} \neq 0$ for some o_0-d_0. There exists an o_1-d_1 s.t. $\Delta_{o_0d_0}\Delta_{o_1d_1} < 0$ and $\Delta_{o_0d_0} + \Delta_{o_1d_1} < 0$.
- We repeat this argument and get an infinite sequence $|\Delta_{o_0d_0}| < |\Delta_{o_1d_1}| < \cdots < |\Delta_{o_jd_j}| < \cdots$.
- Contradiction with finiteness.

Step 2:

There is an arc of D belonging to at least three o-d paths

$\Downarrow$

There exist classes K, increasing costs $(c_a^k(\cdot))$, and demands $(b^{od,k})$ leading to two equilibria with distinct flows

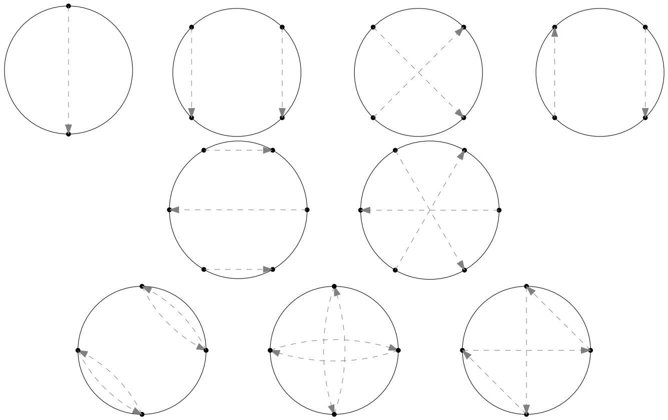
An arc in 3 routes: explicitly building of cost functions and demands leading to two equilibria with distinct flows.

Some features:

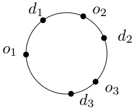
- Three classes.
- Affine cost functions.
- Explicit construction of two equilibria.
- These equilibria are “single-path”.

Structural characterization

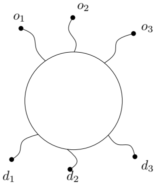
Each arc in at most two *o*-*d* paths $\Leftrightarrow$ $(G, \mathcal{L})$ homeomorphic to a minor of one of



Corollary for general graphs: examples

If 

is in $(G, \mathcal{L})$, G does not have the UP.

If 

is in $(G, \mathcal{L})$, G does not have the UP.

Having a minor without the uniqueness property

A subgraph of $(G, \mathcal{L})$ does not have the UP $\Longrightarrow$ $(G, \mathcal{L})$ does not have the UP.

A minor of $(G, \mathcal{L})$ does not have the UP:

- If the contractions involve only bridges, G does not have the UP.
- If the counterexamples are obtained via "single-path" flows, G does not have the UP.
- And in general, open question.

Strong uniqueness property

G has the strong uniqueness property (SUP) $=$ $(G, \mathcal{L})$ has the UP for any collection of *o*-*d* pairs $\mathcal{L}$

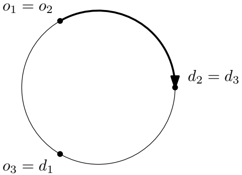
Theorem (M., Pradeau, 2013)

G has the SUP $\iff$ *No cycles of length* 3 *or more.*

Graph having the SUP are thus graphs obtained from a forest by replicating some edges.

Proof

($\Rightarrow$)
The graph



has one arc in three *o*-*d* paths: no UP.

($\Leftarrow$) results from two easy statements:

- A graph with two vertices and parallel edges has the SUP.
- Glueing two graphs on a vertex maintains the SUP.

$\square$

Equivalence of equilibria

Equivalence of equilibria

Two equilibria are equivalent if the contribution of each $(od, k) \in \mathcal{L} \times K$ to the flow on each arc is the same in all equilibria.

Theorem (M.,Pradeau, 2013)

A ring has the uniqueness property $\iff$ *Generically, for every partition of the population into classes, all equilibria are equivalent.*

Uniqueness property: a combinatorial sufficient condition for a 'two-sided' game

Consider a nonatomic congestion game with player-specific cost functions, not necessarily played on a graph.

Proposition

Suppose that there are finite sets A^+ and A^- such that every player i has exactly two available strategies r_i^+ and r_i^- with $r_i^+ \subseteq A^+$ and $r_i^- \subseteq A^-$. Then, if all triples of pairwise distinct strategies have an empty intersection, the uniqueness property holds.

Open questions

Generalization
UP for general graphs?

Minor
What if a minor does not have UP?

Question
What if *one-way* edges are allowed?

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Thank you